Scalar Fields and Vector fields
Definition

• A scalar field is an assignment of a scalar to each point in a region in the space. E.g. the temperature at a point on the earth is a scalar field.

• A vector field is an assignment of a vector to each point in a region in the space. E.g. the velocity field of a moving fluid is a vector field as it associates a velocity vector to each point in the fluid.
Definition

- A scalar field is a map from D to $\mathbb{R}$, where D is a subset of $\mathbb{R}^n$.
- A vector field is a map from D to $\mathbb{R}^n$, where D is a subset of $\mathbb{R}^n$.
- For $n=2$: vector field in plane,
- for $n=3$: vector field in space
- Example: Gradient field
**Definition of Vector Field**

Let $M$ and $N$ be functions of two variables $x$ and $y$, defined on a plane region $R$. The function $\mathbf{F}$ defined by

$$\mathbf{F}(x, y) = Mi + Nj$$

is called a **vector field over $R$**.

Let $M$, $N$, and $P$ be functions of three variables $x$, $y$, and $z$, defined on a solid region $Q$ in space. The function $\mathbf{F}$ defined by

$$\mathbf{F}(x, y, z) = Mi + Nj + Pk$$

is called a **vector field over $Q$**.
Line integral

- Line integral in a scalar field
- Line integral in a vector field
A rescue team follows a path in a danger area where for each position the degree of radiation is defined. Compute the total amount of radiation gathered by the rescue team along the path.
Piecewise Smooth Curves
Piecewise Smooth Curves

A classic property of gravitational fields is that, subject to certain physical constraints, the work done by gravity on an object moving between two points in the field is independent of the path taken by the object. One of the constraints is that the path must be a piecewise smooth curve. Recall that a plane curve $C$ given by

$$r(t) = x(t)i + y(t)j, \quad a \leq t \leq b$$

is smooth if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous on $[a, b]$ and not simultaneously 0 on $(a, b)$. 
Piecewise Smooth Curves

Similarly, a space curve $C$ given by

$$r(t) = x(t)i + y(t)j + z(t)k, \quad a \leq t \leq b$$

is **smooth** if $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$

are continuous on $[a, b]$ and not simultaneously 0 on $(a, b)$.

A curve $C$ is **piecewise smooth** if the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which $C$ is smooth.
Example 1 – *Finding a Piecewise Smooth Parametrization*

Find a piecewise smooth parametrization of the graph of $C$ shown in Figure.
Example 1 – Solution

Because C consists of three line segments $C_1$, $C_2$, and $C_3$, you can construct a smooth parametrization for each segment and piece them together by making the last $t$-value in $C_i$ correspond to the first $t$-value in $C_{i+1}$, as follows.

\[
\begin{align*}
C_1: \quad &x(t) = 0, \quad y(t) = 2t, \quad z(t) = 0, \quad 0 \leq t \leq 1 \\
C_2: \quad &x(t) = t - 1, \quad y(t) = 2, \quad z(t) = 0, \quad 1 \leq t \leq 2 \\
C_3: \quad &x(t) = 1, \quad y(t) = 2, \quad z(t) = t - 2, \quad 2 \leq t \leq 3
\end{align*}
\]
Example 1 – Solution

So, $C$ is given by

$$
r(t) = \begin{cases} 
2t \mathbf{j}, & 0 \leq t \leq 1 \\
(t - 1) \mathbf{i} + 2 \mathbf{j}, & 1 \leq t \leq 2. \\
i + 2 \mathbf{j} + (t - 2) \mathbf{k}, & 2 \leq t \leq 3
\end{cases}$$

Because $C_1$, $C_2$, and $C_3$ are smooth, it follows that $C$ is piecewise smooth.
Parametrization of a curve induces an **orientation** to the curve.

For instance, in Example 1, the curve is oriented such that the positive direction is from $(0, 0, 0)$, following the curve to $(1, 2, 1)$. 

**Piecewise Smooth Curves**
Line Integrals

You will study a new type of integral called a **line integral**

\[ \int_C f(x, y) \, ds \]

Integrate over curve \( C \).

for which you integrate over a piecewise smooth curve \( C \).

To introduce the concept of a line integral, consider the mass of a wire of finite length, given by a curve \( C \) in space.

The density (mass per unit length) of the wire at the point \((x, y, z)\) is given by \( f(x, y, z) \).
Line Integrals

Partition the curve $C$ by the points $P_0, P_1, \ldots, P_n$ producing $n$ subarcs, as shown in Figure.
The length of the $i$th subarc is given by $\Delta s_i$.

Next, choose a point $(x_i, y_i, z_i)$ in each subarc.

If the length of each subarc is small, the total mass of the wire can be approximated by the sum

$$\text{Mass of wire} \approx \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta s_i.$$ 

If you let $||\Delta||$ denote the length of the longest subarc and let $||\Delta||$ approach 0, it seems reasonable that the limit of this sum approaches the mass of the wire.
Line Integrals

Definition of Line Integral

If $f$ is defined in a region containing a smooth curve $C$ of finite length, then the line integral of $f$ along $C$ is given by

$$\int_C f(x, y) \, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \, \Delta s_i$$

or

$$\int_C f(x, y, z) \, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \, \Delta s_i$$

provided this limit exists.
To evaluate a line integral over a plane curve $C$ given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, use the fact that

$$ds = \|\mathbf{r}'(t)\| \, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.$$ 

A similar formula holds for a space curve.
Line Integrals

**THEOREM  Evaluation of a Line Integral as a Definite Integral**

Let $f$ be continuous in a region containing a smooth curve $C$. If $C$ is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \leq t \leq b$, then

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.$$  

If $C$ is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \leq t \leq b$, then

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.$$  

Note that if $f(x, y, z) = 1$, the line integral gives the arc length of the curve $C$. That is,

$$\int_C 1 \, ds = \int_a^b \|\mathbf{r}'(t)\| \, dt = \text{length of curve } C.$$
Example 2 – Evaluating a Line Integral

Evaluate \[ \int_C (x^2 - y + 3z) \, ds \]

where C is the line segment shown in Figure.
Example 2 – *Solution*

Begin by writing a parametric form of the equation of the line segment:

\[ x = t, \quad y = 2t, \quad \text{and} \quad z = t, \quad 0 \leq t \leq 1. \]

Therefore, \(x'(t) = 1, \quad y'(t) = 2, \quad \text{and} \quad z'(t) = 1,\) which implies that

\[
\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.
\]
Example 2 – *Solution*

So, the line integral takes the following form.

\[
\int_C (x^2 - y + 3z) \, ds = \int_0^1 (t^2 - 2t + 3t)\sqrt{6} \, dt
\]

\[
= \sqrt{6} \int_0^1 (t^2 + t) \, dt
\]

\[
= \sqrt{6} \left[ \frac{t^3}{3} + \frac{t^2}{2} \right]_0^1
\]

\[
= \frac{5\sqrt{6}}{6}
\]
Line Integrals

For parametrizations given by \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \), it is helpful to remember the form of \( ds \) as

\[
\text{ds} = \| \mathbf{r}'(t) \| \, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.
\]
• Just as for an ordinary single integral, we can interpret the line integral of a positive function as an area.

• In fact, if $f(x, y) \geq 0$, $\int_C f(x, y) \, ds$ represents the area of one side of the “fence” or “curtain” shown here, whose:

  – Base is $C$.

  – Height above the point $(x, y)$ is $f(x, y)$. 
• Now, let $C$ be a piecewise-smooth curve.

– That is, $C$ is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of $C_{i+1}$ is the terminal point of $C_i$. 
Then, we define the integral of \( f \) along \( C \) as the sum of the integrals of \( f \) along each of the smooth pieces of \( C \):

\[
\int_C f(x, y) \, ds = \int_{C_1} f(x, y) \, ds + \int_{C_2} f(x, y) \, ds + \cdots + \int_{C_n} f(x, y) \, ds
\]
A ship sails from an island to another one along a fixed route. Knowing all the sea currents, how much fuel will be needed?
One of the most important physical applications of line integrals is that of finding the work done on an object moving in a force field.

For example, Figure shows an inverse square force field similar to the gravitational field of the sun.
Line Integrals of Vector Fields

To see how a line integral can be used to find work done in a force field $\mathbf{F}$, consider an object moving along a path $C$ in the field, as shown in Figure.

At each point on $C$, the force in the direction of motion is $(\mathbf{F} \cdot \mathbf{T})\mathbf{T}$. 

**Figure 15.13**
Line Integrals of Vector Fields

To determine the work done by the force, you need consider only that part of the force that is acting in the same direction as that in which the object is moving.

This means that at each point on $C$, you can consider the projection $\mathbf{F} \cdot \mathbf{T}$ of the force vector $\mathbf{F}$ onto the unit tangent vector $\mathbf{T}$.

On a small subarc of length $\Delta s_i$, the increment of work is

$$\Delta W_i = (\text{force})(\text{distance}) \approx [\mathbf{F}(x_i, y_i, z_i) \cdot \mathbf{T}(x_i, y_i, z_i)] \Delta s_i$$

where $(x_i, y_i, z_i)$ is a point in the $i$th subarc.
Line Integrals of Vector Fields

Consequently, the total work done is given by the following integral.

\[ W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds \]

This line integral appears in other contexts and is the basis of the following definition of the line integral of a vector field.

Note in the definition that

\[ \mathbf{F} \cdot \mathbf{T} \, ds = \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\| \, dt = \mathbf{F} \cdot \mathbf{r}''(t) \, dt = \mathbf{F} \cdot d\mathbf{r}. \]
Definition of the Line Integral of a Vector Field

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve $C$ given by

$$\mathbf{r}(t), \quad a \leq t \leq b.$$ 

The line integral of $\mathbf{F}$ on $C$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

$$= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, dt.$$

Line Integrals of Vector Fields
Example – Work Done by a Force

Find the work done by the force field

$$\mathbf{F}(x, y, z) = -\frac{1}{2}x \mathbf{i} - \frac{1}{2}y \mathbf{j} + \frac{1}{4} \mathbf{k}$$

on a particle as it moves along the helix given by

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$

from the point (1, 0, 0) to (−1, 0, 3π), as shown in Figure.
Example – Solution

Because

\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \]

\[ = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} \]

it follows that \( x(t) = \cos t \), \( y(t) = \sin t \), and \( z(t) = t \).

So, the force field can be written as

\[ \mathbf{F}(x(t), y(t), z(t)) = -\frac{1}{2} \cos t \mathbf{i} - \frac{1}{2} \sin t \mathbf{j} + \frac{1}{4} \mathbf{k}. \]
Example – Solution

To find the work done by the force field in moving a particle along the curve $C$, use the fact that

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$$

and write the following.

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \int_0^{3\pi} \left( -\frac{1}{2} \cos t \mathbf{i} - \frac{1}{2} \sin t \mathbf{j} + \frac{1}{4} \mathbf{k} \right) \cdot \left( -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \right) \, dt
\]

\[
= \int_0^{3\pi} \left( \frac{1}{2} \sin t \cos t - \frac{1}{2} \sin t \cos t + \frac{1}{4} \right) \, dt
\]

\[
= \int_0^{3\pi} \frac{1}{4} \, dt = \frac{1}{4} t \bigg|_0^{3\pi} = \frac{3\pi}{4}
\]
Line Integrals of Vector Fields

For line integrals of vector functions, the orientation of the curve $C$ is important.

If the orientation of the curve is reversed, the unit tangent vector $\mathbf{T}(t)$ is changed to $-\mathbf{T}(t)$, and you obtain

$$
\int_{-c} \mathbf{F} \cdot d\mathbf{r} = -\int_{c} \mathbf{F} \cdot d\mathbf{r}.
$$
Line Integrals in Differential Form
Line Integrals in Differential Form

A second commonly used form of line integrals is derived from the vector field notation used in the preceding section.

If \( \mathbf{F} \) is a vector field of the form \( \mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j} \), and \( C \) is given by \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \), then \( \mathbf{F} \cdot d\mathbf{r} \) is often written as \( M\,dx + N\,dy \).

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt \\
= \int_a^b (M\mathbf{i} + N\mathbf{j}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j}) \, dt \\
= \int_a^b \left( M \frac{dx}{dt} + N \frac{dy}{dt} \right) \, dt \\
= \int_C (M\,dx + N\,dy)
\]
Line Integrals in Differential Form

This differential form can be extended to three variables. The parentheses are often omitted, as follows.

\[ \int_C M \, dx + N \, dy \quad \text{and} \quad \int_C M \, dx + N \, dy + P \, dz \]
Example – *Evaluating a Line Integral in Differential Form*

Let \( C \) be the circle of radius 3 given by

\[
r(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi
\]

as shown in Figure. Evaluate the line integral

\[
\int_C y^3 \, dx + (x^3 + 3xy^2) \, dy.
\]
Example – Solution

Because $x = 3 \cos t$ and $y = 3 \sin t$, you have $dx = -3 \sin t \, dt$ and $dy = 3 \cos t \, dt$. So, the line integral is

$$
\int_C M \, dx + N \, dy
$$

$$
= \int_C y^3 \, dx + (x^3 + 3xy^2) \, dy
$$

$$
= \int_0^{2\pi} \left[ (27 \sin^3 t)(-3 \sin t) + (27 \cos^3 t + 81 \cos t \sin^2 t)(3 \cos t) \right] \, dt
$$

$$
= 81 \int_0^{2\pi} (\cos^4 t - \sin^4 t + 3 \cos^2 t \sin^2 t) \, dt
$$

$$
= 81 \int_0^{2\pi} \left( \cos^2 t - \sin^2 t + \frac{3}{4} \sin^2 2t \right) \, dt
$$
Example – Solution

\[
81 \int_0^{2\pi} \left[ \cos 2t + \frac{3}{4} \left( \frac{1 - \cos 4t}{2} \right) \right] dt = 81 \left[ \frac{\sin 2t}{2} + \frac{3}{8} t - \frac{3}{32} \sin 4t \right]_0^{2\pi} = \frac{243\pi}{4}.
\]
Suppose instead of being a force field, suppose that $\mathbf{F}$ represents the velocity field of a fluid flowing through a region in space. Under these circumstances, the integral of $\mathbf{F} \cdot \mathbf{T}$ along a curve in the region gives the fluid’s flow along the curve.

**DEFINITIONS Flow Integral, Circulation**

If $\mathbf{r}(t)$ is a smooth curve in the domain of a continuous velocity field $\mathbf{F}$, the flow along the curve from $t = a$ to $t = b$ is

$$\text{Flow} = \int_{a}^{b} \mathbf{F} \cdot \mathbf{T} \, ds.$$  

The integral in this case is called a flow integral. If the curve is a closed loop, the flow is called the circulation around the curve.
EXAMPLE: Finding Circulation Around a Circle

Find the circulation of the field \( \mathbf{F} = (x - y)i + xj \) around the circle \( \mathbf{r}(t) = (\cos t)i + (\sin t)j \), \( 0 \leq t \leq 2\pi \).

Solution

On the circle, \( \mathbf{F} = (x - y)i + xj = (\cos t - \sin t)i + (\cos t)j \), and

\[
\frac{d\mathbf{r}}{dt} = (-\sin t)i + (\cos t)j.
\]

\[
\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \frac{\sin^2 t + \cos^2 t}{1} = -\sin t \cos t + 1.
\]

Circulation = \( \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt = \int_0^{2\pi} (1 - \sin t \cos t) \, dt \)

= \left[ t - \frac{\sin^2 t}{2} \right]_0^{2\pi} = 2\pi. \]
Flux Across a Plane Curve

To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve \( C \) in the xy-plane, we calculate the line integral over \( C \) of \( \mathbf{F} \cdot \mathbf{n} \), the scalar component of the fluid’s velocity field in the direction of the curve’s outward-pointing normal vector.

**DEFINITION**  Flux Across a Closed Curve in the Plane

If \( C \) is a smooth closed curve in the domain of a continuous vector field \( \mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \) in the plane and if \( \mathbf{n} \) is the outward-pointing unit normal vector on \( C \), the flux of \( \mathbf{F} \) across \( C \) is

\[
\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n} \, ds.
\]
Notice the difference between flux and circulation: Flux is the integral of the normal component of \( \mathbf{F} \); circulation is the integral of the tangential component of \( \mathbf{F} \).
How to evaluate Flux of $\mathbf{F}$ across $\mathbf{C}$

we choose a smooth parameterization

$$x = g(t), \quad y = h(t), \quad a \leq t \leq b,$$

that traces the curve $\mathbf{C}$ \textit{exactly once as $t$ increases from $a$ to $b$}. We can find the outward unit normal vector $\mathbf{n}$ by crossing the curve’s unit tangent vector $\mathbf{T}$ with the vector $\mathbf{k}$.

- If the motion is clockwise, $\mathbf{k} \times \mathbf{T}$ points outward;
- if the motion is counterclockwise, $\mathbf{T} \times \mathbf{k}$ points outward

We choose: $\mathbf{n} = \mathbf{T} \times \mathbf{k}$
Now,

\[ n = T \times k = \left( \frac{dx}{ds} \mathbf{i} + \frac{dy}{ds} \mathbf{j} \right) \times \mathbf{k} = \frac{dy}{ds} \mathbf{i} - \frac{dx}{ds} \mathbf{j}. \]

If \( \mathbf{F} = M(x, y) \mathbf{i} + N(x, y) \mathbf{j} \), then

\[ \mathbf{F} \cdot n = M(x, y) \frac{dy}{ds} - N(x, y) \frac{dx}{ds}. \]

Hence, \[ \int_C \mathbf{F} \cdot n \, ds = \int_C \left( M \frac{dy}{ds} - N \frac{dx}{ds} \right) \, ds = \oint_C M \, dy - N \, dx. \]

Here the circle on the integral shows that the integration around the closed curve \( C \) is to be in the counterclockwise direction.

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**Calculating Flux Across a Smooth Closed Plane Curve**

(Flux of \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} \) across \( C \)) = \[ \oint_C M \, dy - N \, dx \]

The integral can be evaluated from any smooth parametrization \( x = g(t), y = h(t), \)
\( a \leq t \leq b, \) that traces \( C \) counterclockwise exactly once.
EXAMPLE: Finding Flux Across a Circle

Find the flux of $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the $xy$-plane.

Solution

The parametrization $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \ 0 \leq t \leq 2\pi$.

$$M = x - y = \cos t - \sin t,$$
$$\quad \quad dy = d(\sin t) = \cos t \, dt,$$
$$N = x = \cos t,$$
$$\quad \quad dx = d(\cos t) = -\sin t \, dt,$$

$$\text{Flux} = \int_C M \, dy - N \, dx$$

$$= \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t) \, dt$$

$$= \pi.$$

Note that the flux of $\mathbf{F}$ across the circle is positive, implies the net flow across the curve is outward. A net inward flow would have given a negative flux.
Path Independence

**DEFINITIONS**  Path Independence, Conservative Field
Let \( \mathbf{F} \) be a field defined on an open region \( D \) in space, and suppose that for any two points \( A \) and \( B \) in \( D \) the work \( \int_A^B \mathbf{F} \cdot d\mathbf{r} \) done in moving from \( A \) to \( B \) is the same over all paths from \( A \) to \( B \). Then the integral \( \int \mathbf{F} \cdot d\mathbf{r} \) is path independent in \( D \) and the field \( \mathbf{F} \) is conservative on \( D \).

Under differentiability conditions, a field \( \mathbf{F} \) is conservative iff it is the gradient field of a scalar function \( f \); i.e., iff for \( \mathbf{F} = \nabla f \) some \( f \). The function \( f \) then has a special name.

**DEFINITION**  Potential Function
If \( \mathbf{F} \) is a field defined on \( D \) and \( \mathbf{F} = \nabla f \) for some scalar function \( f \) on \( D \), then \( f \) is called a potential function for \( \mathbf{F} \).
once we have found a potential function $f$ for a field $\mathbf{F}$, we can evaluate all the work integrals in the domain of $\mathbf{F}$ over any path between $A$ and $B$ by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$
Connectivity and Simple Connectivity

• All curves are **piecewise smooth**, that is, made up of finitely many smooth pieces connected end to end.

• The components of $\mathbf{F}$ have continuous first partial derivatives implies that when $\mathbf{F} = \nabla f$ this continuity requirement guarantees that the mixed second derivatives of the potential function $f$ are equal.
**Simple curve:** A curve that doesn’t intersect itself anywhere between its endpoints.

\[ r(a) = r(b) \] for a simple closed curve

*But* \( r(a) \neq r(b) \) when \( a < t_1 < t_2 < b \)
**Simply-connected region:** A simply-connected region in the plane is a connected region \( D \) such that every simple closed curve in \( D \) encloses only points that are in \( D \).

Intuitively speaking, a simply-connected region contains no hole and can’t consist of two separate pieces.

An open **connected** region means that every point can be connected to every other point by a smooth curve that lies in the region. Note that connectivity and simple connectivity are not the same, and neither implies the other. Think of connected regions as being in “one piece” and simply connected regions as not having any “holes that catch loops.”
INDEPENDENCE OF PATH

Suppose $C_1$ and $C_2$ are two piecewise-smooth curves (which are called paths) that have the same initial point $A$ and terminal point $B$. We have

$$\int_{C_1} \nabla f \cdot dr = \int_{C_2} \nabla f \cdot dr$$

Note: $F$ is conservative on $D$ is equivalent to saying that the integral of $F$ around every closed path in $D$ is zero. In other words, the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.
THEOREM \textbf{The Fundamental Theorem of Line Integrals}

1. Let $\mathbf{F} = Mi + Nj + Pk$ be a vector field whose components are continuous throughout an open connected region $D$ in space. Then there exists a differentiable function $f$ such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

if and only if for all points $A$ and $B$ in $D$ the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining $A$ to $B$ in $D$.

2. If the integral is independent of the path from $A$ to $B$, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$
EXAMPLE: Finding Work Done by a Conservative Field

Find the work done by the conservative field

\[ \mathbf{F} = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} = \nabla (xyz) \]

along any smooth curve \( C \) joining the point \( A(-1, 3, 9) \) to \( B(1, 6, -4) \).

Solution

With \( f(x, y, z) = xyz \), we have

\[
\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r}
\]

\[ = f(B) - f(A) \quad \text{Fundamental Theorem} \]

\[ = xyz\Big|_{(1,6,-4)} - xyz\Big|_{(-1,3,9)} \]

\[ = (1)(6)(-4) - (-1)(3)(9) \]

\[ = -24 + 27 = 3. \]
**THEOREM**  
**Closed-Loop Property of Conservative Fields**

The following statements are equivalent.

1. \( \int \mathbf{F} \cdot d\mathbf{r} = 0 \) around every closed loop in \( D \).
2. The field \( \mathbf{F} \) is conservative on \( D \).

Proof that Part 1 \( \Rightarrow \) Part 2

If we have two paths from \( A \) to \( B \), one of them can be reversed to make a loop.

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{r} = 0.
\]
Finding Potentials for Conservative Fields

Component Test for Conservative Fields
Let \( \mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k} \) be a field whose component functions have continuous first partial derivatives. Then, \( \mathbf{F} \) is conservative if and only if

\[
\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.
\]
EXAMPLE: Finding a Potential Function

Show that $F = (e^x \cos y + yz)i + (xz - e^x \sin y)j + (xy + z)k$ is conservative and find a potential function for it.

Solution

$M = e^x \cos y + yz, \quad N = xz - e^x \sin y, \quad P = xy + z$

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

We find $f$ by integrating the equations

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \quad \frac{\partial f}{\partial y} = xz - e^x \sin y, \quad \frac{\partial f}{\partial z} = xy + z.$$
Integrating first equation w.r.t. ‘x’

\[ f(x, y, z) = e^x \cos y + xyz + g(y, z). \]

Differentiating it w.r.t. ‘y’ and equating with the \( \frac{\partial f}{\partial y} \):

\[ -e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y, \]

Computing ‘g’ as a function of ‘y’ gives

\[ f(x, y, z) = e^x \cos y + xyz + h(z). \]

Further differentiating ‘f’ w.r.t. to ‘z’ and equating it with \( \frac{\partial f}{\partial z} \)

\[ xy + \frac{dh}{dz} = xy + z, \quad \text{or} \quad \frac{dh}{dz} = z, \]

Integrating, we have

\[ h(z) = \frac{z^2}{2} + C. \]

Hence,

\[ f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C. \]
**DEFINITIONS**  
**Exact Differential Form**

Any expression \( M(x, y, z) \, dx + N(x, y, z) \, dy + P(x, y, z) \, dz \) is a **differential form**. A differential form is **exact** on a domain \( D \) in space if

\[
M \, dx + N \, dy + P \, dz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy + \frac{\partial f}{\partial z} \, dz = df
\]

for some scalar function \( f \) throughout \( D \).

---

**Component Test for Exactness of** \( M \, dx + N \, dy + P \, dz \)

The differential form \( M \, dx + N \, dy + P \, dz \) is exact if and only if

\[
\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.
\]

This is equivalent to saying that the field \( \mathbf{F} = Mi + Nj + Pk \) is conservative.
EXAMPLE: Showing That a Differential Form Is Exact

Show that \( y \, dx + x \, dy + 4 \, dz \) is exact and evaluate the integral

\[
\int_{(1,1,1)}^{(2,3,-1)} y \, dx + x \, dy + 4 \, dz
\]

over the line segment from \((1, 1, 1)\) to \((2, 3, -1)\).

Solution

\[ M = y, \quad N = x, \quad P = 4 \]

\[
\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.
\]

These equalities tell us that \( y \, dx + x \, dy + 4 \, dz \) is exact, so

\[
y \, dx + x \, dy + 4 \, dz = df
\]

for some function \( f \), and the integral’s value is \( f(2, 3, -1) - f(1, 1, 1) \).
We find $f$ up to a constant by integrating the equations
\[
\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 4.
\]

From the first equation we get
\[f(x, y, z) = xy + g(y, z).\]

The second equation tells us that
\[
\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x, \quad \text{or} \quad \frac{\partial g}{\partial y} = 0.
\]

Hence, $g$ is a function of $z$ alone, and
\[f(x, y, z) = xy + h(z).\]
The third of Equations (4) tells us that

\[ \frac{\partial f}{\partial z} = 0 + \frac{dh}{dz} = 4, \quad \text{or} \quad h(z) = 4z + C. \]

Therefore,

\[ f(x, y, z) = xy + 4z + C. \]

The value of the integral is

\[ f(2, 3, -1) - f(1, 1, 1) = 2 + C - (5 + C) = -3. \]