1: See Lecture Notes - 2 (Page no. 1).

2(b): $V = \mathbb{R}^3(\mathbb{R})$

$W = \{ (a,b,c) \mid a,b,c \in \mathbb{R},\ b=0 \}$

In order to show $W \subseteq V$ is a subspace, enough to show that

1> $0 \in W$

2> $x+y \in W \quad \forall\ x,y \in W$

3> $\alpha x \in W \quad \forall\ \alpha \in \mathbb{R},\ x \in W$.

Now $0 = (\underset{a}{0}, \underset{b}{0}, \underset{c}{0}) \in \mathbb{R}^3$ has $b=0$

Thus $(0,0,0) \in W$

Let $x = (x_1, x_2, x_3)$ & $y = (y_1, y_2, y_3) \in W$ be any

$\Rightarrow x_2 = 0,\ y_2 = 0$ — (1)

$x+y = (x_1+y_1,\ x_2+y_2,\ x_3+y_3)$

from (1) we know $x_2+y_2 = 0$

$\Rightarrow x+y \in W$.

Let $\alpha \in \mathbb{R}$ be any scalar & $x = (x_1, x_2, x_3) \in W$

$\Rightarrow x_2 = 0$ — (2)

Now $\alpha \cdot x = \alpha(x_1, x_2, x_3)$

$= (\alpha x_1, \alpha x_2, \alpha x_3)$

from (2) we know $x_2 = 0$

$\Rightarrow \alpha \cdot x_2 = 0$, Thus $\alpha x \in W$.

Thus W satisfies 1, 2 & 3 & Hence is a subspace of V.

(c).

$W = \{ (a, b, c) \mid a, b, c \in \mathbb{R} \; \& \; a = b = c \}$.

For 0 in $\mathbb{R}^3$, $0 = \underset{a}{(0}, \underset{b}{0}, \underset{c}{0)}$

$\Rightarrow a = b = c = 0$

$\Rightarrow (0, 0, 0) \in W$.

let $x = (x_1, x_2, x_3) \in W$ & $y = (y_1, y_2, y_3) \in W$ be any

$\Rightarrow x_1 = x_2 = x_3$ & $y_1 = y_2 = y_3$ —— (1)

Now $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

using (1) we can see that

$x_1 + y_1 = x_2 + y_2 = x_3 + y_3$

$\Rightarrow x + y \in W$.

Consider $\alpha \in \mathbb{R}$ & $x = (x_1, x_2, x_3) \in W$ any

$\Rightarrow x_1 = x_2 = x_3$

observe that

$x_1 = x_2 \Rightarrow x_1 - x_2 = 0$

$\Rightarrow \alpha(x_1 - x_2) = 0$

$\Rightarrow \alpha x_1 - \alpha x_2 = 0$

$\Rightarrow \alpha x_1 = \alpha x_2$

Similarly $\alpha x_2 = \alpha x_3$

So we have $\alpha x_1 = \alpha x_2 = \alpha x_3$

$\Rightarrow (\alpha x_1, \alpha x_2, \alpha x_3) \in W$

i.e. $\alpha (x_1, x_2, x_3) \in W$

Thus W is a subspace of V.

3 (a):

$V = M_{m,n}(\mathbb{R})$

$W_1 = \{ A \in M_n(\mathbb{R}) \mid A^T = -A \}$ → A^T denotes transpose of A.

= Set of all Anti-symmetric matrices.

$0 \in V$ is zero matrix with all entries equal to zero.

So for $0 = (a_{ij})_{i,j=1}^{n}$ s.t. $a_{ij} = 0 \ \forall \ i,j=1^{n}$

we have $0^T = 0 = -0$

Thus $0 \in W_1$.

Let $A = (a_{ij})_{i,j=1}^{n}$ & $B = (b_{ij})_{i,j=1}^{n} \in W_1$ be any

$\Rightarrow A^T = -A \ , B^T = -B$

i.e. $a_{ij} = -a_{ji}$ & $b_{ij} = -b_{ji} \ \forall \ i \neq j$ —— (1)

$A + B = (a_{ij} + b_{ij})_{i,j=1}^{n}$

Using ~~one~~ (1) we can observe that

$a_{ij} + b_{ij} = -(a_{ji} + b_{ji})$

$\Rightarrow (A+B)^T = -(A+B)$

Thus $A + B \in W_1$.

Now let $\alpha \in \mathbb{R}$ be any Scalar & $A = (a_{ij})_{i,j=1}^{n} \in W_1$ be any

$\Rightarrow A^T = -A$

i.e. $a_{ij} = -a_{ji} \ \forall \ i = j$ —— (2)

$\alpha A = (\alpha a_{ij})_{i,j=1}^{n}$

Using (2) we have $\alpha a_{ij} = \alpha(-a_{ji})$

$= -(\alpha a_{ji})$

$\Rightarrow (\alpha A)^T = -\alpha A$

$\alpha A \in W_1$

So, W_1 satisfies all conditions to be a subspace & hence is a subspace of $M_n(\mathbb{R})$.

---

(4): Let $W_1, W_2 \subseteq V$ be any two subspaces of V

To show: $W_1 \cap W_2$ is also a subspace of V

Since $0 \in W_1$ & $0 \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2$.

Let $x, y \in W_1 \cap W_2$

$\Rightarrow x, y \in W_1$ & $x, y \in W_2$

Since both W_1 & W_2 are subspaces so,

$x+y \in W_1$ & $x+y \in W_2$

$\Rightarrow x+y \in W_1 \cap W_2$.

Assume $\alpha \in \mathbb{R}$ & $x_1 \in W_1 \cap W_2$

$\Rightarrow \alpha x_1 \in W_1$ $\because W_1$ is a subspace &

$\alpha x_1 \in W_2$ $\because W_2$ is a subspace

$\Rightarrow \alpha x_1 \in W_1 \cap W_2$.

Thus $W_1 \cap W_2$ forms a subspace.

(5):

Consider $V = \mathbb{R}^2(\mathbb{R})$

$W_1 = \{(x,0) \mid x \in \mathbb{R}\}$

$W_2 = \{(0,y) \mid y \in \mathbb{R}\}$

It is easy to check that both W_1 & W_2 are subspaces of $V = \mathbb{R}^2(\mathbb{R})$.

$W_1 \cup W_2 = \{(x,y) \in \mathbb{R}^2 \mid \text{either } x=0 \text{ or } y=0\}$

= union of x-axis & y-axis in $\mathbb{R}^2$.

clearly $(1,0) \in W_1 \cup W_2$ & $(0,1) \in W_1 \cup W_2$

however $(1,0) + (0,1) = (1,1) \notin W_1 \cup W_2$

Thus $W_1 \cup W_2$ is not a subspace.

($\Rightarrow$) If $W_1 \subset W_2$ or $W_2 \subset W_1$

$\Rightarrow W_1 \cup W_2 = W_2$ or W_1 which is a subspace in both cases.

Condition for $W_1 \cup W_2$ to be a subspace is necessary:

($\Leftarrow$)

Let W_1, W_2 are subspaces of V & $W_1 \cup W_2$ is also a subspace.

Assume that neither $W_1 \subseteq W_2$ nor $W_2 \subseteq W_1$

$\Rightarrow \exists\ w_1 \in W_1$ s.t. $w_1 \notin W_2$ and

$\exists\ w_2 \in W_2$ s.t. $w_2 \notin W_1$

As $W_1 \cup W_2$ is a subspace, we have

$w_1 + w_2 \in W_1 \cup W_2 \Rightarrow w_1 + w_2 \in W_1$ or W_2

case i> If $w_1 + w_2 \in W_1$

$\Rightarrow w_1 + w_2 - w_1 \in W_1$

$\Rightarrow w_2 \in W_1$ ※ Contradiction as $w_2 \notin W_1$.

case ii> If $w_1 + w_2 \in W_2$

$\Rightarrow w_1 + w_2 - w_2 \in W_2$

$\Rightarrow w_1 \in W_2$ ※ Contradic as $w_1 \notin W_2$.

Thus our assumption is wrong & Hence either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

---

(6): $W_1, W_2 \subseteq V$ are subspaces of V,

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2 \}$$

To prove that $W_1 + W_2$ is a subspace, it is enough to show that $W_1 + W_2$ is closed w.r.t. addition & scalar multiplication.

Closed w.r.t addition

Let $x = w_1 + w_2$ & $y = w_1' + w_2'$ be any two elements of $W_1 + W_2$,

$$x + y = (w_1 + w_2) + (w_1' + w_2')$$

$$= (\underbrace{w_1 + w_1'}_{\in W_1}) + (\underbrace{w_2 + w_2'}_{\in W_2})$$ [using Associativity & Commutativity of + in V]

$\in W_1 + W_2$

Scalar multiblication

Let $\alpha \in \mathbb{R}$, $x = w_1 + w_2 \in W_1 + W_2$ be any

then $\alpha \cdot x = \alpha \cdot (w_1 + w_2)$

$$= \underset{\underset{W_1}{\cap}}{\alpha w_1} + \underset{\underset{W_2}{\cap}}{\alpha w_2}$$

$$\in W_1 + W_2$$

Thus $W_1 + W_2$ forms a subspace of V.

claim: $W_1 + W_2$ is the smallest subspace containing W_1 & W_2.

Let $W \subseteq V$ be a subspace s.t.

$W_1 \subset W$ & $W_2 \subset W$

i.e. $W_1 \cup W_2 \subset W$ ——— (1)

Take any element $x = w_1 + w_2 \in W_1 + W_2$ for some $w_1 \in W_1$, $w_2 \in W_2$

from (1) we see that

$w_1 \in W$, $w_2 \in W$

$\Rightarrow$ $w_1 + w_2 \in W$ as W is a subspace.

$\Rightarrow$ $x \in W$

Since x was any arbitrary element of $W_1 + W_2$

$\Rightarrow$ $W_1 + W_2 \subset W$

i.e. any subspace W of V containing W_1 & W_2 also contains $W_1 + W_2$, thus $W_1 + W_2$ is smallest such subspace.

(f) : a > $V = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ is a function} \}$

$$W = \{ f : f(1) = 0 \}$$

Enough to show W is closed w.r.t addition & scalar multiplication.

let $f, g \in W$ be any

$\Rightarrow$ $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(1) = 0$

$g : \mathbb{R} \to \mathbb{R}$ s.t. $g(1) = 0$

$$\Rightarrow (f+g)(1) = f(1) + g(1)$$
$$= 0 + 0$$
$$= 0$$

$\Rightarrow$ $f + g \in W$.

let $\alpha \in \mathbb{R}$ & $f \in W$ be any, Then $f(1) = 0$

$$(\alpha \cdot f)(1) = \alpha \cdot f(1)$$
$$= \alpha \cdot 0$$
$$= 0.$$

$\Rightarrow$ $\alpha f \in W$

Thus W is a subspace of V.

(9) a:

$V = \mathbb{R}^3(\mathbb{R})$

$v_1 = (1,-1,4)$, $v_2 = (-2,1,3)$, $v_3 = (4,-3,5)$

To check whether $S = \{v_1, v_2, v_3\}$ span $\mathbb{R}^3$, let

$v = (x,y,z) \in \mathbb{R}^3$ any & if possible $\exists$

scalars α, β, γ s.t.

$$v = (x,y,z) = \alpha(1,-1,4) + \beta(-2,1,3) + \gamma(4,-3,5)$$

$$\Rightarrow \left.\begin{aligned} x &= \alpha - 2\beta + 4\gamma \\ y &= -\alpha + \beta - 3\gamma \\ z &= 4\alpha + 3\beta + 5\gamma \end{aligned}\right\} \quad \text{— (1)}$$

Solving system of equations in (1) we get

$$x + y = -\beta + \gamma$$

$$4y + z = 7\beta - 7\gamma = -7(-\beta+\gamma)$$

$$\Rightarrow \quad x + y = -\frac{1}{7}(4y + z)$$

$$\Rightarrow \quad 7x + 7y = -4y - z$$

$$\Rightarrow \quad z = -11y - 7x$$

Thus any $(x,y,z) \in \text{span}(S)$ satisfies

$$z = -11y - 7x$$

Therefore all $(x,y,z) \in \mathbb{R}^2$ s.t. $z \neq -11y - 7x \notin \text{span}(S)$.

i.e. v_1, v_2, v_3 does not span $\mathbb{R}^3$.

for eg. $(1,1,1)$ has $1 \neq -11\times1 - 7\times1 = -18$ i.e. $(1,1,1) \notin \text{span}(S)$.

(b) $V = \mathbb{R}^3$, $v_1 = (1,0,1)$, $v_2 = (0,1,1)$

let $S = \{v_1, v_2\}$

$$\text{span}(S) = \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \{\alpha(1,0,1) + \beta(0,1,1) \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \{(\alpha, \beta, \alpha+\beta) \mid \alpha, \beta \in \mathbb{R}\}.$$

Thus subspace spanned by S is

$$\text{span}(S) = \{(x,y,z) \mid z = x+y, \ x, y \in \mathbb{R}\}.$$

for $w = (1,1,-1)$

$x = 1, y = 1, z = -1$

So, $z = -1 \neq \overset{x}{1} + \overset{y}{1} = 2$

i.e. $(1,1,-1) \notin \text{span}(S)$.

10: $P_2 = \{P(x) \mid P(x)$ is Polynomial of degree $\leq 2\}$

$P(x) \in P_2 \Rightarrow P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ for some $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$

Consider $P_0(x) \equiv 1$ i.e. $P_0(x) = 1 \ \forall x \in \mathbb{R}$

$P_1(x) = x \ \forall x \in \mathbb{R}$

$P_2(x) = x^2 \ \forall x \in \mathbb{R}$

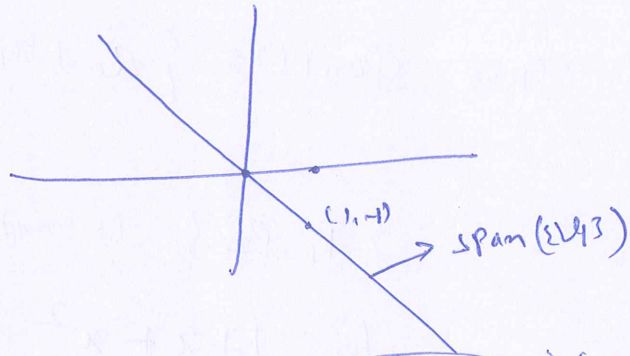
$\Rightarrow P(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \alpha_2 P_2(x)$

i.e. $\{P_0(x), P_1(x), P_2(x)\}$ forms a spanning set of P_2.

(11): $V = \mathbb{R}^2$, $v_1 = (-1,1)$,

$$\text{span}(\{v_1\}) = \{\alpha v_1 \mid \alpha \in \mathbb{R}\}$$
$$= \{\alpha(-1,1) \mid \alpha \in \mathbb{R}\}$$
$$= \{(-\alpha, \alpha) \mid \alpha \in \mathbb{R}\}$$
$$= \{(x,y) \mid y = -x, x \in \mathbb{R}\}$$

Geometrically $\text{span}(\{v_1\})$ is Line $y=-x$ in $\mathbb{R}^2$



---

(12): $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ & $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{R})$

Let $S = \{A_1, A_2, A_3\}$

$$\text{span}(S) = \{\alpha A_1 + \beta A_2 + \gamma A_3 \mid \alpha, \beta, \gamma \in \mathbb{R}\}$$

$$= \left\{\alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ \& } b = c\right\}.$$

(13):

$P_2 = \{ P(x) \mid \deg. p(x) \leq 2 \}$

$P_1(x) = 1+3x, \; P_2(x) = x+x^2$

Let, $S = \{ P_1(x), P_2(x) \}$

$span(S) = \{ \alpha P_1(x) + \beta P_2(x) \mid \alpha, \beta \in \mathbb{R} \}$

$= \{ \alpha(1+3x) + \beta(x+x^2) \mid \alpha, \beta \in \mathbb{R} \}$

$= \{ \alpha + (3\alpha + \beta)x + \beta x^2 \mid \alpha, \beta \in \mathbb{R} \}$

Thus $span(S) = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \; \& \; a_1 = 3a_0 + a_2 \}$.

$\{ P_1, P_2 \}$ is not a spanning set for P_2 as

for $1+x+x^2 \in P_2$

$a_0 = 1$

$a_1 = 1$

$a_2 = 1$

i.e. $a_1 = 1 \neq 3 \cdot \overset{a_0}{1} + \overset{a_2}{1}$

Thus $1+x+x^2 \notin span(S)$ & hence S is not a spanning set for P_2.

(14):

$u_1 = (2,-1), \; u_2 = (3,2)$

Let $(x,y) \in \mathbb{R}^2$ any & assume $\exists \, \alpha, \beta \in \mathbb{R}$ s.t

$(x,y) = \alpha u_1 + \beta u_2$

$= \alpha(2,-1) + \beta(3,2)$

$= (2\alpha + 3\beta, -\alpha + 2\beta)$

$$\Rightarrow \quad \left.\begin{aligned} x &= 2\alpha + 3\beta \\ y &= -\alpha + 2\beta \end{aligned}\right\} \quad (1)$$

Solving system of eqns. in (1) we get

$$\alpha = \frac{1}{7}(2x-3y), \quad \beta = \frac{1}{7}(x+2y). \text{ --- } (2)$$

Thus any $(x,y) \in \mathbb{R}^2$ can be written as $\alpha(2,-1)+\beta(3,2)$

where α, β are given by (2).

In Particular for $(5,-7)$

$$\alpha = \frac{1}{7}(10+21) = \frac{31}{7}$$

$$\beta = \frac{1}{7}(5-14) = \frac{-9}{7}$$

verification: $\frac{31}{7}(2,-1) + \frac{(-9)}{7}(3,2)$

$$= \left(\frac{62-27}{7}, \frac{-31-18}{7}\right)$$

$$= (5,-7).$$

---

(16): $S, S' \subseteq V(\mathbb{R})$ & $S \subseteq S'$

Claim: $\operatorname{Span}(S) \subseteq \operatorname{Span}(S')$.

Let $v \in \operatorname{Span}(S)$ be any vector

$\Rightarrow v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ for some α_i's $\in \mathbb{R}$

v_i's $\in S$

Since $S \subset S'$

$\Rightarrow v_i \in S \subset S'$

$\Rightarrow v_i \in S' \;\forall\; i=1^{K}$

i.e. $v = \alpha_1 v_1 + \cdots + \alpha_k v_k \in \mathrm{Span}(S')$

As v ($\in \mathrm{Span}(S)$) was chosen arbitrary so, we get

$\mathrm{Span}(S) \subseteq \mathrm{Span}(S')$.

(17):

($\Rightarrow$)

Given that $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2\}$

$\Rightarrow v_3 \in \mathrm{Span}\{v_1, v_2\}$

$\Rightarrow v_3 = \alpha v_1 + \beta v_2$ for some $\alpha, \beta \in \mathbb{R}$.

i.e. v_3 is a Linear Combination of v_1, v_2.

($\Leftarrow$)

Assume that v_3 is l.c. of v_1, v_2 i.e.

$$v_3 = \alpha v_1 + \beta v_2 \text{ for some } \alpha, \beta \in \mathbb{R} \qquad (1)$$

To show: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2\}$

Let $v \in \mathrm{Span}\{v_1, v_2, v_3\}$ be any, then

$v = \alpha_1 v_1 + \beta_1 v_2 + \gamma_1 v_3$ for some $\alpha_1, \beta_1, \gamma_1 \in \mathbb{R}$

Using (1) we have

$v = \alpha_1 v_1 + \beta_1 v_2 + \gamma_1(\alpha v_1 + \beta v_2)$

i.e. $v = \underbrace{(\alpha_1 + \gamma_1 \alpha)}_{\in \mathbb{R}} v_1 + \underbrace{(\beta_1 + \gamma_1 \beta)}_{\in \mathbb{R}} v_2$

$\in \text{span}\{v_1, v_2\}$

Thus, $\text{span}\{v_1, v_2, v_3\} \subseteq \text{span}\{v_1, v_2\}$

$\supseteq$ follows from Q(16)

$\Rightarrow \quad \text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2\}.$

18: Similar to Q. no 17, extend from $\{v_1, v_2, v_3\}$ to $\{v_1, v_2, \ldots, v_n\}$.

19 (a): False

Let $V = \mathbb{R}^2 (\mathbb{R})$

$S = \{(1,0), (0,1)\}$ is a spanning set for $\mathbb{R}^2$.

if $W = \{(x,0) / x \in \mathbb{R}\}$, then W is a subspace of V.

As $W \subsetneq \mathbb{R}^2$ so, $W \neq \text{span}(S) = \mathbb{R}^2$

Thus S is not a spanning set for subspace W of V.

19 (C):

False.

let $V = \{ P(x) \mid P(x) \text{ is a Real valued Polynomial} \}$

claim: V has no finite spanning set.

let $S \subseteq V$ be any finite set,

$$m = \max_{P(x) \in S} (\text{degree}(P(x)))$$

i.e. m is max. of degrees over all Polynomials in S.

It can be observed that Span(S) can have only Polynomials of degree less than or equal to m.

Thus $\text{Span}(S) \neq V$, Hence V has no finite spanning set.

(8) Let U be the union of the two coordinate axes in $\mathbb{R}^2$. More precisely, let

$$U = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}.$$

Then clearly U is closed under scalar multiplication.
However, $(1,0)$ & $(0,1)$ are in U but their sum $(1,1)$ is not in U, so U is not closed under addition. Thus U is not a subspace of $\mathbb{R}^2$.

(20) Let $U_1 = \{0\}$, $U_2 = V$ & $W = V$.
Then, $U_1 + W = U_2 + W$ are both equal to V.
but $U_1 \neq U_2$.