Tutorial sheet - 2 Solutions

1 a) $L(S) = \text{span}(S) = \left\{ \sum_{i=1}^{k} \alpha_i v_i \mid v_i \in S, \alpha_i \in \mathbb{R} \right.$

for $\alpha = 1$, $v \in S$ we have

$\alpha \cdot v \in L(S)$

$\Rightarrow 1 \cdot v = v \in L(S) \quad \forall v \in S$

$\Rightarrow S \subseteq L(S)$

Thus $L(S)$ is subspace containing S.

b) $S \subseteq T \subseteq V$, T is a subspace of V th

Let, $v \in L(S) \Rightarrow v = \sum_{i=1}^{k} \alpha_i v_i$ where α

v

As $S \subseteq T$

$\Rightarrow v_i \in T \quad \forall i = 1, k$

$\Rightarrow \sum_{i=1}^{k} \alpha_i v_i \in L(T)$

$\Rightarrow v \in L(T)$

$\Rightarrow L(S) \subseteq L(T)$.

② : $S = \{ A = (a_{ij})_{n \times n} \in M_n(\mathbb{R}) \mid A$ is either a symmetri
an Antisymmetric M

Claim :- S spans $M_n(\mathbb{R})$.

for any $A = (a_{ij})_{n \times n} \in M_n(\mathbb{R})$ define

$$S_A = \frac{A + A^T}{2} \quad \text{and}$$

$$S'_A = \frac{A - A^T}{2}$$

Observe that $S_A^T = \dfrac{(A+A^T)^T}{2} = \dfrac{A^T + (A^T)^T}{2}$

$$= \frac{A^T + A}{2}$$

$$= S_A$$

$\Rightarrow$ S_A is symmetric matrix

$$S_A'^T = \frac{(A - A^T)^T}{2}$$

$$= \frac{A^T - (A^T)^T}{2}$$

$$= \frac{A^T - A}{2}$$

$$= -\frac{A - A^T}{2}$$

$$= -S'_A$$

$\Rightarrow$ S'_A is Anti-symmetric matrix.

Also $S_A + S'_A = \dfrac{A + A^T}{2} + \dfrac{A - A^T}{2}$

$$= A$$

Thus $A = 1 \cdot S_A + 1 \cdot S'_A$ where S_A is symm. & S'_A is Antisymm.

(4):

$u_1 = (1,2,0)$, $u_2 = (-1,1,2)$, $u_3 = (3,0,0)$

Let $v = (a,b,c) \in \mathbb{R}^3$ &

$v = \alpha u_1 + \beta u_2 + \gamma u_3$

$\Rightarrow (a,b,c) = \alpha (1,2,0) + \beta(-1,1,2) + \gamma(3,0,0)$

$$\Rightarrow \left.\begin{aligned} a &= \alpha - \beta + 3\gamma \\ b &= 2\alpha + \beta \\ c &= 2\beta \end{aligned}\right\} \quad (1)$$

Solving system of eqns. in (1), we get

$$\left.\begin{aligned} \alpha &= \frac{1}{4}(2b - c) \\ \beta &= \frac{c}{2} \\ \gamma &= \frac{1}{12}(4a - 2b + 3c) \end{aligned}\right\} \quad (2)$$

Thus any $v = (a,b,c)$ can be written as $\alpha u_1 + \beta u_2 + \gamma u_3$, with α, β & γ given by (2).

5 : (a) $\left\{ \underset{\overset{\downarrow}{A}}{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}}, \underset{\overset{\downarrow}{B}}{\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}} \right\} \subseteq M_2(\mathbb{R})$.

Assume that $\alpha \cdot A + \beta \cdot B = 0$ for som

$$\Rightarrow \alpha \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + \beta \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha - 2\beta & -3\alpha + 6\beta \\ -2\alpha + 4\beta & 4\alpha - 8\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \alpha - 2\beta &= 0 \\ -3\alpha + 6\beta &= 0 \\ -2\alpha + 4\beta &= 0 \\ 4\alpha - 8\beta &= 0 \end{aligned}\right\} \quad (1)$$

Each eqn in (1) gives $\alpha = 2\beta$, so for any $\beta \in \mathbb{R}$, if we take $\alpha = 2\beta$ then

$$\alpha \cdot A + \beta \cdot B = 0$$

for eg. choose $\beta = 1$, so $\alpha = 2 \cdot \beta = 2$

$$2\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + 1\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 2-2 & -6+6 \\ -4+4 & 8-8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\{A, B\}$ is Linearly dependent.

: In fact we can directly see that $B = -2 \cdot A$, so $-2 \cdot A + 1 \cdot B$

(C) $\{(1,-1,2), (2,0,1), (-1,2,-1)\} \subseteq \mathbb{R}^3$
$\quad\quad v_1 \quad\quad v_2 \quad\quad v_3$

Let $\alpha v_1 + \beta v_2 + \gamma v_3 = (0,0,0)$ for some $\alpha, \beta, \gamma \in \mathbb{R}$

$\Rightarrow \quad \alpha(1,-1,2) + \beta(2,0,1) + \gamma(-1,2,-1) = (0,0,0)$

$$\Rightarrow \quad \left.\begin{aligned} \alpha + 2\beta - \gamma &= 0 \\ -\alpha + 0.\beta + 2\gamma &= 0 \\ 2\alpha + \beta - \gamma &= 0 \end{aligned}\right\} \quad (1)$$

Solving system of eqns. in (1) we get

$$\alpha = \beta = \gamma = 0$$

Thus, $\{v_1, v_2, v_3\}$ is L.I. in $\mathbb{R}^3$.

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(6): a) $f(x) = x$, $g(x) = |x|$

Let $\alpha . f(x) + \beta . g(x) = 0$ for some $\alpha, \beta \in \mathbb{R}$.

In Particular for $x = 1$ & -1 we have

$$\alpha . f(1) + \beta . g(1) = 0$$
$$\alpha . f(-1) + \beta\, g(-1) = 0$$

$$\Rightarrow \quad \left.\begin{aligned} \alpha . 1 + \beta . 1 &= 0 \\ -\alpha + \beta &= 0 \end{aligned}\right\} \quad (1)$$

Solving (1) we get $\alpha = \beta = 0$. Thus $\{x, |x|\}$ is L.I. set.

(b) $\{\cos x, \sin x\}$

Let $\alpha \cdot \cos x + \beta . \sin x = 0 \quad \forall x \in \mathbb{R}$ for some $\alpha, \beta \in \mathbb{R}$.

In Particular for $x = 0, \frac{\pi}{2}$ we have,

$$\alpha \cdot \cos 0 + \beta . \sin 0 = 0$$
$$\alpha \cdot \cos \frac{\pi}{2} + \beta . \sin \frac{\pi}{2} = 0$$

$$\Rightarrow \quad \alpha . 1 + \beta . 0 = 0$$
$$\alpha . 0 + \beta . 1 = 0$$

$$\Rightarrow \quad \alpha = 0, \beta = 0$$

Thus $\{\cos x, \sin x\}$ is L.I. set.

(c) $\{e^{rx}, e^{sx}\}, r \neq s$.

Let $\alpha . e^{rx} + \beta . e^{sx} = 0$ for some $\alpha, \beta \in \mathbb{R}$

Choose $x = 0, 1$, we get

$$\alpha . e^{r.0} + \beta . e^{s.0} = 0$$
$$\alpha . e^{r.1} + \beta . e^{s.1} = 0$$

$$\Rightarrow \quad \alpha + \beta = 0 \quad \text{—(1)}$$
$$\alpha . e^{r} + \beta . e^{s} = 0 \quad \text{—(2)}$$

we get $\alpha = -\beta$ from (1), substituting in (2) we get

$$-\beta . e^{r} + \beta e^{s} = 0$$
$$\Rightarrow \beta(-e^{r} + e^{s}) = 0$$

As $r \neq s$ and e^x is one-one

$e^r \neq e^s$ i.e $e^s - e^r \neq 0$

$\Rightarrow \beta = 0$

using in (1), we get $\alpha = 0$

Thus $\{e^{rx}, e^{sx}\}$ for $r \neq s$ is L.I set

x

x

(7.) ~~we have~~ ~~$Av = v$~~

Given information is about v and w vectors

Thus, first we need to relate vectors

$$\begin{bmatrix} -1 \\ 8 \\ -9 \end{bmatrix} \text{ with } v \text{ and } w$$

so let us find linear combination

$$\begin{bmatrix} -1 \\ 8 \\ -9 \end{bmatrix} = av + bw$$

solving this yields $a = 3$ and $b = -2$

Thus,

$$\begin{bmatrix} -1 \\ 8 \\ -9 \end{bmatrix} = 3v - 2w$$

Then we compute

$$A^5\begin{bmatrix}-1\\8\\-9\end{bmatrix} = A^5(3v-2w) = 3A^5v - 2A^5w \quad —(1)$$

Now by assumption we have

$$Av = -v \quad —(2.)$$

using this repeatedly we have $A^5v = (-1)^5 v$
$= -v$

$$(\text{using } Aw = 2w) \quad —(3.)$$

IIIy $A^5w = 2^5 w = 32w$

$$\therefore A^5\begin{bmatrix}-1\\8\\-9\end{bmatrix} = 3(-v) - 2(32w)$$

(from (1.), (2.) & (3.))

$$= -3\begin{bmatrix}1\\2\\-1\end{bmatrix} + 64\begin{bmatrix}2\\-1\\3\end{bmatrix}$$

$$= \begin{bmatrix}-131\\58\\-189\end{bmatrix}$$

(8):

$S = \{(1+i, 2i, 2), (1, 1+i, 1-i)\} \subset \mathbb{C}^3$

Let $\alpha(1+i, 2i, 2) + \beta(1, 1+i, 1-i) = (0,0,0)$, α, β ...

$\Rightarrow (\alpha+\alpha i, 2\alpha i, 2\alpha) + (\beta, \beta+\beta i, \beta-\beta i) = (0,0,...$

$$\Rightarrow \begin{aligned} (\alpha+\beta) + \alpha i &= 0 = 0+0i \\ \beta + (2\alpha+\beta) i &= 0 = 0+0i \\ (2\alpha+\beta) - \beta i &= 0 = 0+0i \end{aligned}$$

Comparing Real & imaginary Parts in above equations we get $\alpha = \beta = 0$.

Thus S is L.I. set in $\mathbb{C}^3$.

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(9).

Observe that

$$\underset{v_1}{(1+i, 2i, 2)} = 1+i \; \underset{v_2}{(1, 1+i, 1-i)}$$

Thus $1 \cdot v_1 - (1+i) \cdot v_2 = (0,0,0)$

i.e. $\{v_1, v_2\}$ is Linearly dependent over $\mathbb{C}$.

V - vector space, $S_1 \subset S_2 \subset V$

(11): a) See Lecture notes 2 (Page no. 12).

b) S_1 is Linearly dependent

$\Rightarrow \exists$ finite no. of distinct vectors $v_1, v_2, \dots, v_n$ & scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ s.t. not all α_i

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$, $\alpha_i \neq 0$ for some i

Since $S_1 \subset S_2$

$\Rightarrow$ v_i for $i = 1, \ldots, n$ also belong to S_2

So we have $\{v_i\}_{i=1}^{n} \subseteq S_2$ & $\alpha_i \in \mathbb{R}$ s.t. $\alpha_i \neq 0$ for some

s.t. $\sum_{i=1}^{n} \alpha_i v_i = 0$

$\Rightarrow$ S_2 is L.D. set in V.

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(12): $S \subset V(\mathbb{R})$

$0 \in S$

Choose any $\alpha \neq 0$, $\alpha \in \mathbb{R}$

Since $\alpha \cdot 0 = 0$ $\forall \alpha \in \mathbb{R}$

$\Rightarrow$ S contains a vector namely 0 & there exists non zero scalar α s.t.

$\alpha \cdot 0 = 0$

Hence S is Linearly dependent.

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(13): $S \subset V$. S is L.I., $v \notin S$.

($\Rightarrow$) Let, $S \cup \{v\}$ is L.I.

Claim: $v \notin \mathrm{span}(S)$

Suppose on contrary if $v \in \mathrm{span}(S)$

$\Rightarrow$ $\exists$ vectors $v_1, v_2, \ldots, v_k \in S$ & scalars α_i's $\in \mathbb{R}$ s.t.

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$

$$\Rightarrow \quad 1\cdot v - \alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_k v_k = 0$$

Clearly $1 \neq 0$ & $v, v_1, v_2, \ldots, v_k \in S \cup \{v\}$

Thus $S \cup \{v\}$ is L.D. $\Rightarrow \Leftarrow$ Contradicts our assumption $S \cup \{v\}$ is L.I.

$(\Leftarrow)$ Let $v \notin \text{span}(S)$

Claim:- $S \cup \{v\}$ is L.I.

Assume that $S \cup \{v\}$ is not L.I., i.e. $S \cup \{v\}$ is L.D. so $\exists$ vectors $\{v_i\}_{i=1}^{k}$ in $S \cup \{v\}$ & scalars α_i's$_{i=1}^{k}$ not all zero s.t.

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \quad \text{——} \quad (1)$$

Since S is given to be L.I., so in (1) one of v_i must be equal to v, as otherwise we will have all $\alpha_i = 0$.

Let without loss of generality $v_1 = v$

$\Rightarrow$ $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$ is equivalent to,

$$\alpha_1 v + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

Since $\{v_2, \ldots, v_k\} \subset S$ & S is L.I. so, we have $\alpha_1 \neq 0$

$$\Rightarrow \quad v = -\alpha_1^{-1}(\alpha_2 v_2 + \cdots + \alpha_k v_k) \in \text{Span}(S)$$

which contradicts $v \notin \text{span}(S)$.

Thus $S \cup \{v\}$ is L.I.

(17): $V = M_n(\mathbb{R})$

a) $W = \{ A = (a_{ij})_{n\times n} \mid A \text{ is diagonal} \}$

$$A \in W \Rightarrow A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$$

$$\Rightarrow A = a_{11}\begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & \ddots \; 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & & \\ & 1 & 0 \\ 0 & & 0 \ddots \; 0 \end{bmatrix} + \cdots + a_{nn}\begin{bmatrix} 0 & & \\ & 0 & 0 \\ & 0 \ddots & \\ & & 1 \end{bmatrix}$$

$$\Rightarrow A \in \text{span}\{\underbrace{\underset{1}{\begin{bmatrix} 1 & & 0 \\ & 0 \ddots & \\ 0 & & 0 \end{bmatrix}}, \underset{2}{\begin{bmatrix} 0 & & \\ & 1 & 0 \\ 0 & \ddots & 0 \end{bmatrix}}, \cdots\cdots, \underset{n}{\begin{bmatrix} 0 & & \\ & 0 & 0 \\ 0 & \ddots & 1 \end{bmatrix}}}_{\text{denoted by } S}\}$$

So W is spanned by the set S containing n matrices.

To show S is a basis of W, it is enough to show S is L.I.

Let
$$D_i = \begin{array}{c c} & i \\ & \downarrow \\ i \rightarrow & \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 1 & \\ 0 & & & \ddots \; 0 \end{bmatrix}_{n\times n} \end{array} ; \quad a_{ii} = 1, \; a_{ij} = 0 \text{ for all other } ij\text{'}.$$

if $\alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_n D_n = 0$

$$\Rightarrow \alpha_1 \begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & \ddots \ 0 \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\text{zero Mat}}.$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \alpha_3 & \\ 0 & & & \ddots \ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & \ddots \ 0 \end{bmatrix}$$

$$\Rightarrow \quad \alpha_i = 0 \quad \forall \ i = {}_1^n.$$

Thus S is L.I. set & hence forms a basis of W.

$$\Rightarrow \quad \dim(W) = n.$$

(b) $W = \{ A = (a_{ij})_{n\times n} \mid A^T = -A \}$, all Anti-symmetric matrices

$A \in W \Rightarrow A = (a_{ij})_{n\times n}$ s.t.

$$a_{ii} = 0 \quad \forall \ i = {}_1^n$$

$$a_{ij} = -a_{ji} \quad \forall \ i \neq j.$$

$$A = \begin{bmatrix} 0 & a_{12} & \cdots & a_{j1} & \cdots & a_{1n} \\ -a_{12} & 0 & & & & \\ \vdots & & 0 & & & \\ -a_{1j} & & & \ddots & & \\ \vdots & & & & & \\ -a_{1n} & \cdots & & & & 0 \end{bmatrix}$$

$$= a_{12} \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 \end{bmatrix} + \cdots + a_{1n} \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & & \\ \vdots & & \ddots & 0 \\ -1 & 0 & & 0 \end{bmatrix} + \cdots$$

$$+ a_{23}\begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & +1 & O \\ 0 & -1 & 0 & \\ & O & & \ddots & 0 \end{bmatrix} + \cdots + a_{2n}\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 \cdots & & 1 \\ \vdots & & 0 & & \\ & & & \ddots & \\ 0 & -1 & & & 0 \end{bmatrix} + \cdots$$

$$+ a_{n-1\,n}\begin{bmatrix} 0 & & & \\ & 0 & & O \\ O & & \ddots & \\ & & 0 & 1 \\ 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Denote by

$$\underset{\substack{i<j \\ i,j=1}}{A_{ij}}^{n} = \begin{array}{c} \\ \\ i^{th} \\ j^{th} \\ \\ \end{array}\begin{array}{c} \begin{array}{cc} i^{th} & j^{th} \end{array} \\ \begin{bmatrix} 0 & & 0 & 0 & \\ & 0 & 0 & \vdots & O \\ 0 & \cdots & 0 & \cdots 1 & \\ 0 & 0 \cdots & -1 & \cdots 0 & \\ & O & & & 0 \end{bmatrix} \end{array} \quad \begin{array}{l} a_{ij}=1,\ a_{ji}=-1 \\ a_{ij}=0 \text{ otherwise} \end{array}$$

$$S = \{A_{ij} \mid i<j,\ i,j=1^{n}\}$$

Thus, every Anti-symm. matrix $A \in \operatorname{span}(S)$.

We show that S is L.i. set;

Let $\displaystyle\sum_{\substack{i<j \\ i,j=1}}^{n} \alpha_{ij} A_{ij} = 0$

$$\Rightarrow \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ -\alpha_{12} & 0 & \alpha_{23} & \cdots & \alpha_{2n} \\ -\alpha_{31} & -\alpha_{23} & 0 & & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ & & & \ddots & \alpha_{n-1\,n} \\ -\alpha_{1n} & -\alpha_{n2} & \cdots & -\alpha_{n-1n} & 0 \end{bmatrix} = \begin{bmatrix} & & \\ & O & \\ & & \end{bmatrix}$$

zero matrix

$$\Rightarrow \alpha_{ij} = 0 \ \forall\ i<j,\ i,j=1^{n}.$$

Thus S is L.I. set & Hence forms a Basis of W.

$$\Rightarrow \dim(W) = \frac{n(n-1)}{2}.$$

(d) $W = \{ A = (a_{ij})_{n\times n} \mid A \text{ is upper triangular} \}$

$A \in W \Rightarrow A = (a_{ij})_{n\times n}$ s.t. $a_{ij} = 0 \ \forall\, i > j$.

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & a_{33} & \vdots \\ & 0 & \ddots & a_{n-1\,n} \\ & & & a_{nn} \end{bmatrix}$$

i.e. $$A = a_{11}\begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & \ddots \\ & & & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & 0 \cdots 0 \\ & 0 & 0 \\ 0 & & \ddots \\ & & & 0 \end{bmatrix} + \cdots + a_{1n}\begin{bmatrix} 0 & 0 \cdots & 1 \\ & 0 & 0 \\ 0 & & \ddots \\ & & & 0 \end{bmatrix}$$

$$+ a_{22}\begin{bmatrix} 0 & 0 & \\ & 1 & 0 \\ 0 & & 0 \\ & & & \ddots \\ & & & & 0 \end{bmatrix} + \cdots + a_{2n}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ & 0 & \cdots & 1 \\ 0 & & \ddots & 0 \\ & & & 0 \end{bmatrix} + \cdots$$

$$\cdots + a_{n-1\,n}\begin{bmatrix} 0 & & 0 \\ & 0 & \\ 0 & & \ddots & 1 \\ & & & 0 \end{bmatrix} + a_{nn}\begin{bmatrix} 0 & & 0 \\ & 0 & \\ 0 & & \ddots \\ & & & 1 \end{bmatrix}.$$

Denote for $i, j = 1^n$ s.t. $i \geq j$, $$A_{ij} = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & & & j\text{th} & & \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & \\ & \ddots & & \vdots & & 0 \\ 0 & \cdots & 0 & \cdots 1 & & \vdots \\ & & & & & \vdots \\ & 0 & & & & 0 \end{bmatrix} \end{array}$$ i.e. $a_{ij} = 1$, all other entries $= 0$

Thus any $A \in W$ belongs to $\text{span}(S)$ where
$$S = \{ A_{ij} \mid i \geq j,\ i, j = 1^n \}.$$

Thus S spans W, Now we show S is L.I.,

Assume $$\sum_{\substack{i \geq j \\ i,j=1}}^{n} \alpha_{ij} A_{ij} = 0$$

$$\Rightarrow \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ & \alpha_{22} & & \vdots \\ & & \ddots & \\ & 0 & \alpha_{ii} \cdots & \alpha_{in} \\ & & & \vdots \\ & & & \alpha_{nn} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

zero matrix

$$\Rightarrow \quad \alpha_{ij} = 0 \quad \forall \; i \geq j, \; i,j = 1, \dots, n$$

Thus S is L.I. set & Hence forms a basis of W.

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(18) :

Consider $\mathbb{C} = \{x+iy \mid x,y \in \mathbb{R}\}$ as $\mathbb{R}$ & $\mathbb{C}$ vector space.

Claim: $\dim \mathbb{C}(\mathbb{R}) = 2$

$\dim \mathbb{C}(\mathbb{C}) = 1$

Case 1. $\mathbb{C}(\mathbb{R})$

$S = \{1, i\}$ span $\mathbb{C}$ as for any $x+iy \in \mathbb{C}$

$x+iy = x \cdot 1 + y \cdot i \in \operatorname{span}(\{1, i\})$

Also, for $\alpha, \beta \in \mathbb{R}$ if:

$\alpha \cdot 1 + \beta \cdot i = 0 = 0 + 0i$

$\Rightarrow \alpha = 0, \beta = 0$ i.e. S is L.I. Hence $\dim \mathbb{C}(\mathbb{R}) = 2$

Case 2- $\mathbb{C}(\mathbb{C})$

$S = \{1\}$ forms a basis of $\mathbb{C}$ over $\mathbb{C}$ as vector space.

$$\text{Span}(S) = \{\alpha \cdot 1 \mid \alpha \in \mathbb{C}\}$$
$$= \mathbb{C}$$

Also, S is L.i, since $\alpha \cdot 1 = 0$ for $\alpha \in \mathbb{C}$

$$\Rightarrow \alpha = 0$$

Thus $\dim \mathbb{C}(\mathbb{C}) = 1$.

i.e. $\mathbb{C}$ as a vector space over $\mathbb{R}$ has dimension 2 times the dimension of $\mathbb{C}$ considered as vector space over $\mathbb{C}$.

<u>Is this always true</u> (for finite dimensional vector spaces)

Yes, if V is both a $\mathbb{R}$ vector space & $\mathbb{C}$ vector space then $\dim V(\mathbb{R}) = 2 \times \dim V(\mathbb{C})$.

if $\dim V(\mathbb{C}) = n$, say $\{v_1, v_2, \dots, v_n\}$ is a basis of V over $\mathbb{C}$.

It is easy to check that $\{v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n\}$ forms a basis of V as $\mathbb{R}$ vector space.

Hence $\dim V(\mathbb{R}) = 2n$.

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(19): a) $S = \{P(x) \in P_n(\mathbb{R}) \mid P(0) = 0\}$

for any $P(x) = a_0 + a_1 x + \dots + a_n x^n \in S$

we have $P(0) = 0$

$$\Rightarrow a_0 = 0$$

i.e. $P(x) = a_1 x + a_2 x^2 + \dots + a_n x^n$

$\Rightarrow \quad P(x) = a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n \in \text{Span}(\{x, x^2, \ldots, x^n\})$

let $S' = \{x, x^2, \ldots, x^n\}$

claim S' is L.I.

if $\alpha_1 x^1 + \alpha_2 x^2 + \cdots + \alpha_n x^n = 0$ for some α_i's $\in \mathbb{R}$

$\Rightarrow \quad \alpha_i = 0 \ \forall \ i = 1, \ldots, n$

Thus S' forms a basis of S & $\dim(S) = n$.

(b): $S = \{P(x) \in P_n(\mathbb{R}) \mid P(x) \text{ is an odd function}\}$

$P(x) \in S \Rightarrow P(-x) = -P(x)$

if $P(x) = a_0 + a_1 x + \cdots + a_n x^n$

$\Rightarrow \quad a_0 + a_1(-x) + a_2(-x)^2 + \cdots + a_n(-x)^n = -(a_0 + a_1 x + \cdots + a_n x^n)$

Comparing coefficients on both sides we get

$a_i = -a_i$ for i even

$\Rightarrow 2a_i = 0$

$\Rightarrow a_i = 0$ for all even $i \in \{1, 2, \ldots, n\}$

i.e. $P(x) = \sum_{i=1}^{n} a_i x^i$; $a_i = 0$ for i even.

Case. 1 $\quad n$ is even

let, $S' = \{x, x^3, \ldots, x^{n-1}\}$

$\Rightarrow \quad P(x) \in \text{Span}(S')$

It is easy to check that S' is L.I., Hence forms a basis of S.

i.e. $\dim S = \frac{n}{2}$ when n is even.

Case 2: n is Odd

let $S'' = \{x, x^3, \ldots, x^n\}$, we have

$P(x) \in \text{Span}(S'')$

S'' can also be easily checked to be L.I.,

thus forms a basis of S.

$\Rightarrow \dim(S) = \frac{n+1}{2}$ when n is Odd.

(c) $S = \{P(x) \in P_n(\mathbb{R}) \mid P(0) = P''(0) = 0\}$

$P(x) = a_0 + a_1 x + \cdots + a_n x^n \in S$

$\Rightarrow P(0) = 0$ & $P''(0) = 0$

$P(0) = 0$ gives $a_0 = 0$

$P''(0) = 0 \Rightarrow a_2 = 0$

i.e. $P(x) = a_1 x + a_3 x^3 + \cdots + a_n x^n$

let $S' = \{x, x^3, \ldots, x^n\}$

$\Rightarrow P(x) \in \text{Span}(S')$

Also S' is a L.I. set so, forms a basis of S.

$\Rightarrow \dim S = n-1$.

(20) : $V =$ vector space of all real polynomials over $\mathbb{R}$

Suppose if possible V is finite dimensional

Let $B = \{ P_1(x), P_2(x), \dots, P_t(x) \}$ forms a basis of V

denote by $K = \max_{i=1}^{t} \text{degree}(P_i(x))$

i.e. K is max. among degree of all polynomials $P_i(x)$ for $i = {}_1^t$.

Now any linear combination of $P_i(x)$ say,

$P(x) = \alpha_1 P_1(x) + \dots + \alpha_t P_t(x)$ can have max degree K as mult. by scalars do not ~~[illegible]~~ increase the power of x in $P_i(x)$.

Thus for $P(x) \in \text{span}(B)$

$$\deg(P(x)) \leq K.$$

i.e. no polynomial $P(x)$ with $\deg(P(x)) > K$ can be spanned by polynomials in B, so B is not a spanning set & Hence not a basis of V

$\Rightarrow$ V is not finite dimensional vector space.

$\Rightarrow$ V is not finite dimensional vector space.

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(21) b>

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 \mid a_2 = a_3 = a_4,\ a_4 + a_5 = 0\}.$$

for any $v = (a_1, a_2, a_3, a_4, a_5) \in W_2$ we have,

$$a_2 = a_3 = a_4 \ \&\ a_1 + a_5 = 0 \Rightarrow a_1 = -a_5 \text{ or } a_5 = -a_1$$

Thus v is of the form

$$v = (a_1, a_2, a_2, a_2, -a_1)$$
$$= a_1(1,0,0,0,-1) + a_2(0,1,1,1,0)$$

for $a_1, a_2 \in \mathbb{R}$.

$$\Rightarrow v \in \text{span}(\underbrace{\{(1,0,0,0,-1), (0,1,1,1,0)\}}_{S}).$$

claim: $S = \{(1,0,0,0,-1), (0,1,1,1,0)\}$ is L.I.

Let $\alpha(1,0,0,0,-1) + \beta(0,1,1,1,0) = (0,0,0,0,0)$

$$\Rightarrow (\alpha, \beta, \beta, \beta, -\alpha) = (0,0,0,0,0)$$

$$\Rightarrow \alpha = 0, \beta = 0$$

Thus S spans W_2 & is L.I., Hence forms a basis of W_2.

$$\Rightarrow \dim(W_2) = 2.$$

x — End — x

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(14) Note that vectors v_1, v_2 and v_3 are linearl independent if and only if the matrix

$$A = [v_1\ v_2\ v_3] = \begin{bmatrix} 1 & h & 1 \\ 0 & 1 & 2h \\ 0 & -h & 3h+1 \end{bmatrix}$$

is non-singular

Also, the matrix A is non-singular iff $\det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 1 & h & 1 \\ 0 & 1 & 2h \\ 0 & -h & 3h+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2h \\ -h & 3h+1 \end{vmatrix}$$

$$= 2h^2 + 3h - 1$$

$$= (2h+1)(h+1)$$

ence, $\det(A) \neq 0$ iff $h \neq -\frac{1}{2}, -1$

Therefore, the vectors v_1, v_2, v_3 are L.I for any values of h except $\frac{-1}{2}$ and -1

x x

16. We know that $(1,0)$ & $(0,1)$ forms basis of $\mathbb{R}^2$ over $\mathbb{R}$

Thus, to show given set forms a basis we only have to show 2 vectors are linearly independent

Consider

$$\alpha(1,4) + \beta(0,1) = 0 \quad ; \alpha, \beta \in \mathbb{R}$$

$$(\alpha, 4\alpha + \beta) = (0,0)$$

$$\Rightarrow \alpha = 0, \beta = 0$$

H.P

Q.23 $(a,b,c,d) \in V \iff b-2c+d=0$

$\iff (a,b,c,d) = (a,b,c,2c-b)$

$= \cancel{a(0,0,0,}$

$= (a,0,0,0) + (0,b,0,-b) + (0,0,c,2c)$

$= a(1,0,0,0) + b(0,1,0,-1) + c(0,0,1,2)$

This shows that every vector in V is a linear combination of three linearly independent vectors $(1,0,0,0)$, $(0,1,0,-1)$ and $(0,0,1,2)$

Thus basis of V is $A = \{(1,0,0,0), (0,1,0,-1), (0,0,1,2)\}$

Hence, $\dim V = 3$

Further
$(a,b,c,d) \in W \Rightarrow a=d,\ b=2c$

$\iff (a,b,c,d) = (a,2c,c,a)$

$= (a,0,0,a) + (0,2c,c,0)$

$= a(1,0,0,1) + c(0,2,1,0)$

$\therefore$ Basis for $W =\ B = \{(1,0,0,1), (0,2,1,0)\}$

$\therefore \dim W = 2$

Next, $(a,b,c,d) \in V \cap W \iff (a,b,c,d) \in V$ and $(a,b,c,d) \in W$

$\iff b-2c+d=0,\ a=d,\ b=c$

$\iff (a,b,c,d) = (0,2c,c,0) = c(0,2,1,0)$

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Thus, basis of $V \cap W = \{(0,2,1,0)\}$

and $\dim V \cap W = 1$

$\dim (V+W) = \dim V + \dim (W) - \dim (V \cap W)$

$= 3 + 2 - 1 = 4$

Since $V + W$ is subspace of $\mathbb{R}^4$ and both have same dimension

$\therefore \mathbb{R}^4 = V + W$

x x

(22) Let X be arbitrary column vector of length n and let 0 be zero vector of same size

The columns of M are linearly independent iff the only soln to eq^n $MX = 0$ is the vector $X = 0$

Consider upper Δar matrix

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ 0 & m_{2,2} & m_{2,3} \cdots & m_{2,n} \\ 0 & 0 & m_{3,3} \cdots & m_{3,n} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 \cdots & m_{n,n} \end{bmatrix}$$

$\because M$ is upper Δar, we can use back substitution to conc.

The bottom row gives $\Rightarrow$ $m_{n,n} x_n = 0$

$m_{n,n} \neq 0 \quad \therefore x_n = 0$

Second to last row gives $\quad m_{n-1,n-1}(x_{n-1}) + m_{n-1,n}(x_n) = 0$

$\because x_n = 0$ & $m_{n-1,n-1} \neq 0$

$\Rightarrow x_{n-1} = 0$

continue working backward we get

$$x_i = 0 \ \forall i$$

Thus $X = 0$ $\therefore$ column vectors of M are L.I

further, for second part

Consider $M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$