①

a) $T: \mathbb{R}^3 \to \mathbb{R}^2, \; T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} xy \\ yz \end{bmatrix}$

$$T\left(\alpha\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \beta\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = T\begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{pmatrix} = \underbrace{\begin{bmatrix} (\alpha x_1 + \beta x_2)(\alpha y_1 + \beta y_2) \\ (\alpha y_1 + \beta y_2)(\alpha z_1 + \beta z_2) \end{bmatrix}}_{①}$$

$$\alpha T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \beta T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{bmatrix} \alpha x_1 y_1 \\ \alpha y_1 z_1 \end{bmatrix} + \begin{bmatrix} \beta x_2 y_2 \\ \beta y_2 z_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 y_1 + \beta x_2 y_2 \\ \alpha y_1 z_1 + \beta y_2 z_2 \end{bmatrix} \; ②$$

$\because$ ① $\neq$ ② $\therefore$ T is not Linear.

b) $\because T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\therefore$ T is not linear.

c) $T(\alpha A + \beta B) = (\alpha A + \beta B)^t = \alpha A^t + \beta B^t = \alpha T(A) + \beta T(B)$

$\therefore$ T is linear.

d) Let $A = (a_{ij})_{n\times n}$; $B = (b_{ij})_{n\times n}$

$$\Rightarrow tr(\alpha A + \beta B) = \sum_{i=1}^{n}(\alpha a_{ii} + \beta b_{ii}) = \alpha \sum_{i=1}^{n} a_{ii} + \beta \sum_{i=1}^{n} b_{ii}$$

$$= \alpha\, tr(A) + \beta\, tr(B)$$

$\therefore$ T is linear.

e) Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$q(x) = b_0 + b_1 x + \cdots + b_n x^n$.

$$E_u(\alpha p(x) + \beta q(x))$$

$$= E_u((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + \cdots + (\alpha a_n + \beta b_n)x^n)$$

$$= (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)u + \cdots + (\alpha a_n + \beta b_n)u^n$$

$$= (\alpha a_0 + \alpha a_1 u + \cdots + \alpha a_n u^n) + (\beta b_0 + \beta b_1 u + \cdots + \beta b_n u^n)$$

$$= \alpha E_u(P(x)) + \beta E_u(q(x))$$

$\therefore$ E_u is linear.

f) $T(\alpha p(x) + \beta q(x))$

$$= (\alpha a_0 + \beta b_0, \alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ldots, \alpha a_n + \beta b_n, \ldots)$$

$$= \alpha(a_0, a_1, \ldots, a_n, \ldots) + \beta(b_1, b_2, \ldots, b_n)$$

$$= \alpha T(p(x)) + \beta T(q(x))$$

$\therefore$ T is linear.

(2) (a) False $\because$ all constant functions map to zero.
$\Rightarrow$ $\ker D \neq \{0\}$
$\therefore D$ is not injective.

(b) False $\because (1,1,1,1,1,1,\dots)$ is not in the range of T. ($\because$ polynomials are of finite degree)

(c) True $\because E_4(x-u+a) = a$ for any $a \in \mathbb{R}$

(d) False $\mathrm{Tr}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -(n-1) \end{pmatrix} = 0$ or Trace of all matrices whose diagonal entries add upto zero are in the kernel of Trace mapping.

$\therefore \ker \mathrm{Tr} \neq \{0\}$
$\therefore \mathrm{Tr}$ is not injective.

3. (a) Null space$(S) = \{X \in \mathbb{R}^n : S(X) = 0\}$ where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$

$\Rightarrow$ if $x \in$ Null space (S), then $\begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = 0, x_2 = 0, \dots, x_{n-1} = 0$

$\Rightarrow$ every element of Null space (S) of the form $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{pmatrix}$ for some $x_n \in \mathbb{R}$

$\therefore$ Null space $(S) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix} \in \mathbb{R}^n : x_n \in \mathbb{R} \right\}$.

Range space $(S) = \{Y \in \mathbb{R}^n : S(X) = Y\}$

$= \left\{ \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in \mathbb{R}^n : x_1, \dots, x_{n-1} \in \mathbb{R} \right\}$.

(b) if $X \in$ Null space(T)

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 - x_1 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 0 = x_2 = \dots = x_{n-1} = x_n$

$\Rightarrow$ Null space $= \{0\}$.

$\Rightarrow$ Nullity$(T) = 0$ $\Rightarrow$ Rank$(T) = n$ [using Rank nullity theorem]

$\Rightarrow$ Range space$(T) = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, \dots a_{n-1}, a_n \in \mathbb{R} \right\}$.

(4)

Let us assume that

$v, Tv, \dots T^{m-1}v$ are L.D

$\Rightarrow \alpha_1 v + \alpha_2 Tv + \dots + \alpha_m T^{m-1} v = 0$

& let k be the 1st index such that $\alpha_k \neq 0$

$\Rightarrow T^{m-k}(\alpha_1 v + \alpha_2 Tv + \dots \alpha_m T^{m-1} v) = 0$

$\Rightarrow \alpha_k T^{m-1} v = 0$

$\Rightarrow T^{m-1} v = 0$. ※

a contradiction.

$\therefore v, Tv, \dots T^{m-1}v$ are L.I.

(5)

if $p(x) \in$ Kernel(T)

$\Rightarrow \begin{bmatrix} P(1)-P(2) & 0 \\ 0 & P(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow P(0) = 0$

$P(1) = P(2)$.

Let $p(x) = a_0 + a_1 x + a_2 x^2$

$P(0) = 0 \Rightarrow a_0 = 0$

$P(1) = P(2) \Rightarrow a_1 + a_2 = 2a_1 + 4a_2$

$\Rightarrow a_1 = -3a_2$

$\therefore P(x) = a_2(x^2 - 3x)$, $a_2 \in \mathbb{R}$

$\therefore$ Ker(T) = Span$\{(x^2-3x)\}$

$\therefore$ nullity(T) = dim(Ker T) = 1.

$\Rightarrow$ By Rank-nullity theorem, Rank$(T) = 3-1 = 2$

$\because$ dim$(P_2(x)) = 3$.

Ans.

(7) Let $\beta = \{(1,1), (1,0)\}$ be a basis of $\mathbb{R}^2$

we will find linear operator satisfying

$T(1,1) = (1,3)$

$T(1,0) = (0,0)$ → for singularity condition

Let (x,y) be arbitrary element of $\mathbb{R}^2$

$(x,y) = \alpha(1,1) + \beta(1,0)$

$= y(1,1) + (x-y)(1,0)$

$T(x,y) = y\,T(1,1) + (x-y)\,T(1,0)$

$= y(1,3).$

$\Rightarrow T(x,y) = (y, 3y)$

$\Rightarrow T(1,0) = (0,0)$

$T(0,1) = (1,3)$

$\Rightarrow B = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ Ans.

(9)

Range space is spanned by $\{(1,2,0,-4),(2,0,-1,-3)\}$

Consider $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \\ -4 & -3 & 0 \end{bmatrix}_{4\times 3}$

$T_A : \mathbb{R}^3 \to \mathbb{R}^4$

$T_A(x_1, x_2, x_3) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \\ -4 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$R(T_A) = \text{Colspace}(A)$
$= \text{span}\{(1,2,0,-4),(2,0,-1,-3)\}$

So $T_A(x_1, x_2, x_3) = (x_1 + 2x_2, 2x_1, -x_2, -4x_1 - 3x_2)$.

(10)

(a) $T: \mathbb{R}^2 \to \mathbb{R}^2$
s.t. $T(x,y) = (y,x)$
$\Rightarrow N(T) = \{0\}, \; R(T) = \mathbb{R}^2$

$U: \mathbb{R}^2 \to \mathbb{R}^2$
s.t. $U(x,y) = (-y,x)$
$\Rightarrow N(U) = \{0\}, \; R(U) = \mathbb{R}^2$

(b) $T(x_1, x_2) = (x_2, 0)$

$N(T) = \{(x_1, x_2) : (x_2, 0) = (0,0)\}$
$= \{(x_1, 0) : x_1 \in \mathbb{R}\}$
$= \text{span}\{(1,0)\}$

$R(T) = \text{span}\{T(1,0), T(0,1)\}$
$= \text{span}\{(0,0),(1,0)\}$
$= \text{span}\{(1,0)\}$

$\therefore N(T) = R(T)$

(12) Given that ST is identity operator.

$\Rightarrow$ ST is one-one and onto.

$\because$ $\ker T \subseteq \ker ST = \{0\}$

$\Rightarrow$ $\ker T = \{0\}$

$\therefore$ T is one-one.

again $\operatorname{Im} ST \subseteq \operatorname{Im} S$

$\because$ ST is onto $\Rightarrow$ S is onto

Hence Both S & T are bijective

$\because$ maps from finite dimensional vector space V. to V.

Again $(TS)^2 = (TS)(TS) = T(ST)S = TS$

$\therefore$ TS is Identity.

Counter example:

Let $V = \mathbb{R}^\infty$

Define $S(c_1, c_2, \cdots) = (c_2, c_3, \cdots)$

$T(c_1, c_2, \cdots) = (0, c_1, \cdots)$

clearly ST is identity operator, while TS is not.

(13)

$T: \mathbb{R}^3 \to \mathbb{R}^2 \quad S: \mathbb{R}^2 \to \mathbb{R}^3$

$\Rightarrow ST: \mathbb{R}^3 \to \mathbb{R}^3$

$\because \ker T \subseteq \ker ST$

$\because$ nullity of T is atleast 1.

$\Rightarrow$ nullity of ST atleast 1.

$\therefore$ ST is not injective.

(14) Basis for Im (T):

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \quad \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So col space (A) $= \text{span}\{(1,1,3), (2,3,8)\}$

$\because \{(1,1,3), (2,3,8)\}$ is L.I set

$\Rightarrow$ Im (T) has the basis $\{(1,1,3), (2,3,8)\}$

$\therefore \dim(R(A)) = 2 = \text{Rank}(T)$.

Basis for null-space of T :-

$N(T) = \{x \in \mathbb{R}^4 : Ax = 0\}$

$$\because \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 + 2x_3 - 3x_4 = 0$

$x_1 + 2x_2 + 3x_3 + x_4 = 0$

let $x_3 = k_1,\ x_4 = k_2$

$\left.\begin{array}{l} \Rightarrow x_2 = -2k_1 + 3k_2 \\ x_1 = k_1 - 7k_2 \end{array}\right\} \Rightarrow \quad \text{Null}(A) = \left\{ k_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} : k_1, k_2 \in \mathbb{R} \right\}$

$\Rightarrow$ $S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of Null space (A).

$\therefore$ Nullity $(T) = 2$

(9)

(11) Let $(x,y,z) \in \mathbb{R}^3$.

Then $(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$

$\Rightarrow T(x,y,z) = (x-y)\,T(1,0,0) + (y-z)\,T(1,1,0) + z\,T(1,1,1)$

$= (x-y)(1,0,0) + (y-z)(1,1,1) + z(1,1,0)$

$= (x, y, y-z).$

Let $x = (a,b,c) \in \operatorname{Ker} T$

$\Rightarrow T(x) = (a, b, b-c) = (0,0,0)$

$\Rightarrow \operatorname{Ker} T = \{(0,0,0)\}$

$R(T) = \{(x,y,y-z) : x,y,z \in \mathbb{R}\}$

$= \operatorname{Span}\{(1,0,0), (0,1,1), (0,0,-1)\}$

$\therefore T^3((x,y,z)) = T^2(x,y,\cancel{yz}) = T(x,y,y-(y-z))$

$= T(x,y,z).$

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(8) (a) $\because$ T is linear

$\Rightarrow$ rank T + Nulity T = dim V

$\Rightarrow$ rank T = dim V − Nulity T

$\leq$ dim V.

Given that, dim V < dim W,

So, rank T < dim W $\Rightarrow$ $R(T) \neq W$

$\therefore$ T can not be onto.

(b) $\because$ nulity(T) = dim(V) − rank(T) $\geq$ dim(V) − dim(W) > 0

$\Rightarrow$ nulity T $\neq 0$

$\Rightarrow$ T is not one-one.

(15) $\because \{(1,1),(2,3)\}$ is a basis of $\mathbb{R}^2$

$$T(8,11) = T\{2(1,1)+3(2,3)\}$$
$$= 2\cdot T(1,1) + 3\cdot T(2,3)$$
$$= 2\,(1,0,2) + 3\,(1,1,4)$$
$$= (2,0,4) + (3,3,12)$$
$$= (5,3,16)$$

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(6) .

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(6) Reflection through x-axis is given by

$$T_1(x,y) = (x,-y)$$

Reflection through the line $y=x$ is given by

$$T_2(x,y) = (y, x)$$

Thus on combining these two

$$T = T_2 \circ T_1$$

$$T(x,y) = T_2 \circ T_1 (x,y)$$

$$= T_2 (x,-y)$$

$$= (-y, x).$$