

(1)

Q. 1

Let $T(x, y) = (x', y')$

$$y = r \sin \theta, \quad x = r \cos \theta$$

$$\begin{aligned} x' &= r \cos(\theta + 45^\circ) \\ &= r \cos \theta \cos 45^\circ + (-r \sin \theta \sin 45^\circ) \end{aligned}$$

ATQ

$$\begin{aligned} \Rightarrow x' &= \frac{1}{\sqrt{2}} (r \cos \theta - r \sin \theta) \\ &= \frac{1}{\sqrt{2}} (x - y) \end{aligned}$$

$$\text{Similarly } y' = \frac{1}{\sqrt{2}} (x + y)$$

$$T(x, y) = \frac{1}{\sqrt{2}} (x - y, x + y)$$

let $\beta = \{(1, 0), (0, 1)\}$

$$T((1, 0)) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$T((0, 1)) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow \beta' = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}$$

which is L.T

$\Rightarrow$ Basis of $\mathbb{R}^2$.

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} (1, 0) + \frac{1}{\sqrt{2}} (0, 1)$$

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{2}} (1, 0) + \frac{1}{\sqrt{2}} (0, 1)$$

$$[\phi: \beta', \beta] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ say}$$

② (b), Let (α, β) be the coordinates of the point $(5, 6)$ under the given rotation,

$$\Rightarrow (5, 6) = \alpha \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) + \beta \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\alpha + \beta = 6\sqrt{2}$$

$$\alpha - \beta = 5\sqrt{2}$$

$$\Rightarrow \alpha = \frac{11}{2}\sqrt{2}, \quad \beta = \frac{1}{2}\sqrt{2}$$

$$\Rightarrow (\alpha, \beta) = \left(\frac{11}{2}\sqrt{2}, \frac{1}{2}\sqrt{2} \right)$$

Q.2 (a) $T(x, y) = (2x - 3y, x + 4y)$

$$E = \{(1, 0), (0, 1)\}$$

$$S = \{(1, 3), (2, 5)\}$$

To find $[T: E, S]$

$$T(1, 0) = (2, 1)$$

$$= \alpha(1, 3) + \beta(2, 5)$$

$$= -8(1, 3) + 5(2, 5)$$

$$T(0, 1) = (-3, 4) = 23(1, 3) - 13(2, 5)$$

$$\therefore [T: E, S] = \begin{bmatrix} -8 & 23 \\ 5 & -13 \end{bmatrix} = A$$

(b) $T(1, 3) = (-7, 13)$

$$= -7(1, 0) + 13(0, 1)$$

$$T(2, 5) = (-11, 22)$$

$$= -11(1, 0) + 22(0, 1)$$

(3)

$$[T: S, E] = \begin{bmatrix} -7 & -11 \\ 13 & 22 \end{bmatrix} = B$$

$$(C) \quad \begin{array}{ccc} E & \xrightarrow{A} & S \\ P \downarrow I_1 & & I_2 \downarrow Q \\ S & \xrightarrow{B} & E \end{array}$$

I_1 & I_2 are identity map.

$$T \circ I_1 : E \rightarrow E$$

$$I_2 \circ T : E \rightarrow E$$

$$\text{ie } BP = QA \Rightarrow Q^{-1}B = AP^{-1}$$

$$[P: E, S] = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$[Q: S, E] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Q.3. $B = \{ P_1(x) = 1+x, P_2(x) = x+x^2, P_3(x) = x^2+x^3, P_4(x) = x+x^3 \}$

$$DP_1(x) = 1 = \underset{1+x}{P_1} - \underbrace{B \left(\frac{P_2 + P_4 - P_3}{2} \right)}_{x}$$

$$\begin{aligned} DP_2(x) = 1+2x &= \left[\cancel{P_1} - \cancel{B \left(\frac{\cancel{P_2} + \cancel{P_4} - \cancel{P_3}}{2} \right)} \right] 1+x + 0x \\ &= P_1 + \frac{1}{2} (P_2 + P_4 - P_3) = \end{aligned}$$

$$DP_3(x) = 2x + x^2 \cdot 3 = 2 \left(\frac{P_2 + P_4 - P_3}{2} \right) + 3 \left(\frac{P_2 + P_3 - P_4}{2} \right) = \frac{5P_2 + P_3}{2} - \frac{P_4}{2}$$

$$\begin{aligned} DP_4(x) = 1+3x^2 &= 1+x-x+3x^2 \\ &= \cancel{P_1} - \left(\frac{P_2 + P_4 - P_3}{2} \right) + 3 \left(\frac{P_2 + P_3 - P_4}{2} \right) = P_1 + P_2 + 2P_3 - 2P_4 \end{aligned}$$

$$[D]_{\beta} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1/2 & 1/2 & 5/2 & 1 \\ 1/2 & -1/2 & 1/2 & 2 \\ -1/2 & 1/2 & -1/2 & -2 \end{bmatrix}$$

Ans

Q.4

Referring to sub-4, Qu. 4

if $v \in V$

Then one can easily show that

$\{v, (T-dI)v, (T-dI)^2v, \dots, (T-dI)^{n-1}v\}$ are n LI vectors. if $(T-dI)^m v = 0$, $(T-dI)^m v \neq 0$ $m < n$

$\Rightarrow$ They form a basis for V .

Now Taking $\{w_1 = (T-dI)^{n-1}v, w_2 = (T-dI)^{n-2}v, \dots, w_n = v\} = \beta$

Now to calculate $[T]_{\beta}$

We have $(T-dI)^n v = 0$ (given)

$$(T-dI)w_1 = 0$$

$$\Rightarrow Tw_1 = dw_1$$

$$= d w_1 + 0 w_2 + \dots + 0 w_n \quad \text{--- (1)}$$

11/8

$$(T-dI)w_2 = w_1$$

$$T(w_2) = d w_2 + w_1 + 0 w_3 + 0 \dots + 0 w_n \quad \text{--- (2)}$$

$$T(w_i) = 0 \cdot w_1 + 0 \cdot w_2 + \dots + 1 \cdot w_{i-1} + \dots + 0 \cdot w_n \quad (3)$$

$$i = 3 \dots n$$

n-2 eq's

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$

Proved

Q.5 We are given that matrix representⁿ w.r.t. every basis is same.

$\Rightarrow$ let β_1, β_2 be two basis of V .

$$\Rightarrow [T]_{\beta_1} = [T]_{\beta_2} = A$$

let say P is the change of matrix from β_1 to β_2

$$\Rightarrow [T]_{\beta_1} = P^{-1} [T]_{\beta_2} P$$

$$\Rightarrow A = P^{-1} A P \Rightarrow \boxed{PA = AP}$$

Since β_1, β_2 are arbitrary $\Rightarrow P$ arbitrary

$\therefore A$ commutes with all invertible matrices of order $(\dim(V) \times \dim(V))$

✳ Since every matrix can be written as sum of two invertible matrix,

✳ $\Rightarrow A$ commutes with all matrix,

$\Rightarrow PA = AP$, P is an arbitrary matrix,

⊗

Let $V \in V$, $AV \in V$.

Here we need to prove that V, AV are L.D.

By contradiction, suppose V, AV are L.I.

$\Rightarrow P_1 = \{V, AV\}$ can be extended to a basis of V , $B = \{V, AV, u_3, u_4, \dots, u_n\}$

Now we define a mapping P st

$$P(V) = V, \quad P(AV) = V$$

$$P(u_i) = 0, \quad i = 3(1)n.$$

$$\Rightarrow P(AV) = V$$

$$\Rightarrow (PA)V = V$$

$$\Rightarrow (AP)V = V$$

$$\Rightarrow A(PV) = V$$

$$\Rightarrow AV = V.$$

$\Rightarrow V, AV$ are Linearly dependent

$$\Rightarrow A V = \alpha V,$$

7

• let $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V .

$$\Rightarrow A v_i = \alpha_i v_i$$

$$\text{let } v \in V, \Rightarrow v = \sum_{i=1}^n B_i v_i$$

$$\Rightarrow \cancel{AV} = A \left(\sum_{i=1}^n B_i v_i \right) = \alpha \left(\sum_{i=1}^n B_i v_i \right)$$

$$\Rightarrow \sum_{i=1}^n B_i (A v_i) = \alpha \sum_{i=1}^n B_i (\alpha v_i)$$

$$\Rightarrow \sum_{i=1}^n B_i (\alpha_i v_i) = \sum B_i (\alpha v_i)$$

• Now we compare the coefficient of

v_i

$$\Rightarrow \alpha_i = \alpha, \quad \forall i = 1(1)n,$$

$$\Rightarrow A = \alpha I,$$

Q.6 Since it is given that $\text{Rank}(T) = 1$

$\Rightarrow \exists$ a basis B , st $[T]_B$ has $(n-1)$ columns zero (assuming $\dim V = n$)

& lower diagonal entry is non zero,

i.e. $\text{trace } [T]_B$

$$\therefore [T]_B = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \text{trace}(T) \end{bmatrix}$$

$$I + T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 + \text{tr}(T) \end{bmatrix}$$

$$\det[I + T] = 1 + \text{tr}(T)$$

Hence proved

(7)

(b)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

eigen value of A is $\lambda = 1$.

$$d_1 = d_2 = 1.$$

eigen vector ~~x~~ (let $x = (x_1, x_2)^T$) corresponds to $d_1 = 1$.

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = x_1$$

$$x_2 = 0$$

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A does not have two LI eigenvectors.

$\Rightarrow A$ is not diagonalizable.

Q.9 (a) F, if x_1 is an eigenvector, then $2x_1$ is also an eigenvector. (10)

(b) F, If d_1 & d_2 are eigen value; x_1, x_2 are corresponding eigenvector.

~~$$Ax_1 = d_1 x_1, \quad Ax_2 = d_2 x_2$$~~

~~$$\Rightarrow A(x_1 + x_2) = (d_1 + d_2)$$~~

~~$$\Rightarrow \exists Ax = (d_1 + d_2)x$$~~

In Identity matrix, $d_1 = 1, d_2 = 1$
 $d = d_1 + d_2 \neq 2$.

(c) The sum of two eigenvectors is also an eigenvector if both the eigenvectors are L.D.

F.

(d) T. (by theorem).

(e) F, If A is matrix, and P is any invertible matrix of same order, then

$$B = P^{-1} A P$$

B is similar P .

If $X = [x_1, x_2, \dots, x_n]$ be the eigenvector of

$$\begin{aligned} \text{Then } AX &= DX \\ \Rightarrow AP^{-1}PX &= DX \\ \Rightarrow P(AP^{-1}P)X &= PDX \end{aligned} \quad D = \text{diag}[d_1, d_2, \dots, d_n]$$

$$AX_i = \lambda X_i$$

(11)

$$\Rightarrow A I X_i = \lambda X_i$$

$$\Rightarrow A P^T P X_i = \lambda X_i$$

$$\Rightarrow (P A P^T) P X_i = \lambda P X_i$$

$\Rightarrow P X_i$ are the eigen vector of B .

(f). λ is an eigenvalue of A ,

$$\Rightarrow \det(A - \lambda I) = 0.$$

$$\Rightarrow \det((A + KI) - (\lambda + K)I) = 0$$

$\Rightarrow \lambda + K$ is an eigenvalue of $A + KI$.

Q.10 $A = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(12)

Characteristic eqⁿ is

$\det(A - dI) = 0$ using formula

$$\left\{ \begin{aligned} (a-b)^3 &= a^3 - b^3 - 3a^2b \\ &\quad + 3ab^2 \end{aligned} \right\}$$

$$(2-d)^3 = 0$$

$$\Rightarrow d^3 - 6d^2 + 12d - 8 = 0$$

Characteristic polynomial is

$$d^3 - 6d^2 + 12d - 8 = 0$$

By Cayley Hamilton theorem

since A is invertible as $d \neq 0$

$$A^{-1} [A^3 - 6A^2 + 12A - 8I] = 0$$

$$\Rightarrow A^2 - 6A + 12 = 8A^{-1}$$

$$\Rightarrow A^{-1} = \frac{A^2 - 6A + 12}{8}$$

$$= \frac{1}{8} \left(\begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} - 6 \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} + 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{bmatrix} \text{ Ans.}$$

Q.11

let d be an eigen value of T , x be the corresponding eigen vector.

$$\Leftrightarrow Ax = dx$$

$$\Leftrightarrow (A - dI)x = 0$$

$$\Rightarrow x \in N(A - dI)$$

$$\Leftrightarrow x \in N(T - dI)$$

Q.12, (a) $V = \mathbb{R}^2$, $T(a, b) = (-2a + 3b, -10a + 9b)$

$$T \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} -2 & 3 \\ -10 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Let $B = \{(0, 1), (1, 0)\}$.

$$[T]_B = \begin{bmatrix} -2 & 3 \\ -10 & 9 \end{bmatrix}$$

eigen values of T are given as,

$$\det \begin{pmatrix} -2-d & 3 \\ -10 & 9-d \end{pmatrix} = 0$$

$$\Rightarrow (d-9)(d+2) + 30 = 0$$

$$\Rightarrow d^2 - 7d - 18 + 30 = 0$$

$$\Rightarrow d^2 - 7d + 12 = 0$$

$$d = 3, 4$$

eigen vector corres. to $d=3$,

$$AX = dX$$

$$\Rightarrow \begin{bmatrix} -5 & 3 \\ -10 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -5x_1 + 3x_2 = 0$$

$$\Rightarrow x_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

eigen vector corres to $d=4$

$$\begin{bmatrix} -6 & 3 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-6x_1 + 3x_2 = 0$$

$$2) \quad -5x_1 + x_2 = 0$$

$$x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P' = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

$$\cancel{[T]_{B'}} \quad B' = \{(3, 1), (1, 2)\}?$$

~~Q2~~ (b) $V = \mathbb{P}_2$ and $T\{f(x)\} = x f'(x) + f(2)x + f(3)$

~~$f(x) = 1$~~ , ~~$T(1)$~~ = for basis $B' = \{1, x, x^2\}$

$$T(1) = x \cdot 0 + x + 1 = 1 + x + 0 \cdot x^2$$

$$= \cancel{2 + x + 0x^2}$$

$$T(x) = x \cdot 1 + x \cdot 2 + 3 = 3 + 3x + 0 \cdot x^2$$

$$= \cancel{5 + 2x + 0x^2}$$

$$T(x^2) = x \cdot 2x + x \cdot 4 + 9 = 9 + 4x + 2x^2$$

$$= \cancel{13 + 4x + 2x^2}$$

$$[T]_{B'} = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

eigen value of T will be $\lambda = 2, 1, 9$.

$$[T]_{B=\{1, x, x^2\}} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & 0 \\ 9 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 9 \end{bmatrix}$$

eigen value of T. $\det \begin{bmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 9-\lambda \end{bmatrix}$

$$\Rightarrow (9-\lambda) [(1-\lambda)(3-\lambda) - 3] = 0$$

$$\Rightarrow (9-\lambda) [3 - 4\lambda + \lambda^2 - 3] = 0$$

$$\Rightarrow \lambda = 0, 9, 4.$$

eigen vector corres. to $\lambda = 0$

$$\begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_3 = 0.$$

$$x_1 + x_2 = 0 \Rightarrow x_1 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

$$\boxed{\lambda = 9}$$

$$\begin{bmatrix} -8 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -8x_1 + 3x_2 + 9x_3 = 0 \\ x_1 + x_2 + 4x_3 = 0 \end{cases}$$

$$\Rightarrow 4x_2 + 9x_3 = 0$$

$$\begin{bmatrix} -1 & 3 & 9 \\ 0 & 4 & 13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$4x_2 + 13x_3 = 0$$

$$x_2 = -13/4 x_3$$

$$x = \begin{bmatrix} 3 \\ 13 \\ -4 \end{bmatrix}$$

$$x_1 = 3x_2 + 9x_3$$

$$= 3(-13/4)x_3 + 9x_3 = -3/4 x_3$$

$$d=4.$$

$$\begin{bmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -3 & 3 & 9 \\ 0 & 0 & 13 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_3 = 0.$$

$$x_1 = x_2$$

$$x_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$p = \begin{bmatrix} -3 & -3 & 1 \\ 1 & -13 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

$$p' = \{ -3 + x, -3 - 13x + 4x^2, 1 + x \}$$

$$(c) \quad V = M_{2 \times 2}, \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = e_{22}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e_{12}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = e_{21}$$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = e_{11}, \right.$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e_{12},$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = e_{21},$$

$$\left. \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = e_{22} \right\}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \{e_{11}\}$$

16

$$\Rightarrow [T]_B = \begin{bmatrix} 0 & a & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

eigen value of T is given by $\det(T - dI) = 0$

$$\det \begin{bmatrix} -d & 0 & 0 & 1 \\ 0 & 1-d & 0 & 0 \\ 0 & 0 & 1-d & 0 \\ 1 & 0 & 0 & -d \end{bmatrix} = 0$$

$$(1-d) \begin{vmatrix} -d & 0 & 1 \\ 0 & 1-d & 0 \\ 1 & 0 & -d \end{vmatrix} = 0$$

$$\Rightarrow (1-d) \left[-d(-d(1-d)) + 1[0 - (1-d)] \right] = 0$$

$$\Rightarrow (1-d) [d^2(1-d) - (1-d)] = 0$$

$$\Rightarrow (1-d)^2 (d^2 - 1) = 0$$

$$d = 1, 1, 1, -1$$

~~eigenvalue~~ eigenvector corresponds to $d = 1$

$$AX = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$-x_1 + x_4 = 0 \Rightarrow x_1 = x_4$$

x_2, x_3 are arbitrary

$$(d) \quad V = M_{2 \times 2}, \quad T(A) = A^t - \frac{1}{2} \text{tr}(A) I_2$$

18

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{aligned} T(e_{11}) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= -e_{11} - 2e_{22} \end{aligned}$$

$$T(e_{12}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = e_{21}$$

$$T(e_{21}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e_{12}$$

$$\begin{aligned} T(e_{22}) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -2e_{11} - e_{22} \end{aligned}$$

$$T = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & -1 \end{bmatrix}$$

$$\det(T - dI) = 0$$

$$\det \begin{bmatrix} -1-d & 0 & 0 & -2 \\ 0 & -d & 1 & 0 \\ 0 & 1 & -d & 0 \\ -2 & 0 & 0 & -1-d \end{bmatrix} = 0$$

$$\Rightarrow (-1-d)(-1-d)(d^2-1) + 2[-2(d^2-1)] = 0$$

$$\Rightarrow (d^2-1)[(d+1)^2-4] = 0$$

$$\Rightarrow \cancel{d=1, -1}, (d+1)(d-1)(d+1-2)(d+1+2) = 0$$

$$\Rightarrow d = -1, 1, +1, d = 3$$

eigen vectors corres. to $d = -1$

19

$$\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_4 = x_1 = 0$$

$$x_2 + x_3 = 0$$

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$d = 1$

$$\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -2 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 + x_4 = 0$$

$$x_2 = x_3$$

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$d = -3$

$$\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 = x_4$$

$$3x_2 + x_3 = 0$$

$$x_1 + 3x_3 = 0$$

$$x_1 = x_3 = 0$$

$$x_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$B' = \{ e_{12} - e_{21}, e_{12} + e_{21}, e_{11} - e_{22}, e_{11} + e_{22} \}$$

$$= \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Ans.

$$T\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(17)

eigenvector corresponds to $\lambda = -1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_2 = 0, \quad x_3 = 0 \\ x_1 + x_4 = 0$$

$$x_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$$B = \{e_{11} + e_{22}, e_{12}, e_{21}, e_{11} - e_{12}\}$$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$