MA 102: Linear Algebra & Integral Transforms
Tutorial Sheet - 1
Second Semester of the Academic Year 2019-2020

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1. Using only the defining axioms of a vector space V over a field $\mathbb{F}$, prove the following properties of V:
   (a) $0v = \mathbf{0}$ for every $v \in V$.
   (b) $a\mathbf{0} = \mathbf{0}$ for every $a \in \mathbb{F}$.
   (c) $(-1)v = -v$ for every $v \in V$.

2. For the vector space $V = \mathbb{R}^3$ over $\mathbb{R}$, check whether $W \subseteq V$ as given below, is a subspace or not:
   (a) $W = \{(a, b, c) : a, b, c \in \mathbb{R} \mid a + b + c = 1\}$.
   (b) $W = \{(a, b, c) : a, b, c \in \mathbb{R} \mid b = 0\}$.
   (c) $W = \{(a, b, c) : a, b, c \in \mathbb{R} \mid a = b = c\}$.

3. Let $V = \mathbb{M}_{m,n}(\mathbb{R})$ be the vector space containing all $m \times n$ matrices with entries in $\mathbb{R}$. Then,
   (a) for $m = n$, prove that the set $W_1 \subseteq V$ consisting of all antisymmetric matrices forms a subspace of V.
   (b) for $m = n$, show that the set $W_1 \subseteq V$ of all matrices with $trace(M) = 0$ for all $M \in W_1$ is a subspace of V.

4. Prove that the intersection $W_1 \cap W_2$ of two subspaces $W_1, W_2 \subseteq V$ is again a subspace of V.

5. Give examples of subspaces $W_1, W_2 \subseteq V$ of a vector space V such that $W_1 \cup W_2$ is not a subspace of V. Observe that if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ then $W_1 \cup W_2$ is always a subspace of V, is the condition necessary also?

6. Let $W_1, W_2 \subseteq V$ be two subspaces of V, define $W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$. Prove that $W_1 + W_2$ is a subspace of V, also establish that it is the smallest subspace of V containing both W_1 and W_2.

7. Let V be the vector space of the functions $f : \mathbb{R} \to \mathbb{R}$. Show that W is a subspace of V , where:
   (a) $W = \{f(x) : f(1) = 0\}$, all functions whose value at 1 is 0.
   (b) $W = \{f(x) : f(3) = f(1)\}$, all functions assigning the same value to 3 and 1.
   (c) $W = \{f(x) : f(-x) = -f(x)\}$; the set of odd functions.

8. Give an example of a nonempty subset U of R^2 such that U is close under scalar multiplication, but U is not a subspace of R^2.

9. (a) Determine whether the vectors $v_1 = (1, -1, 4)$, $v_2 = (-2, 1, 3)$ and $v_3 = (4, -3, 5)$ span $\mathbb{R}^3$.
   (b) If $V = \mathbb{R}^3$, $v_1 = (1, 0, 1)$ and $v_2 = (0, 1, 1)$, determine the subspace of $\mathbb{R}^3$ spanned by v_1 and v_2. Does $w = (1, 1, -1)$ lie in this subspace?

10. Determine a spanning set for P_2, the vector space of all polynomials of degree less than or equal to 2.

11. If $V = \mathbb{R}^2$ and $v_1 = (-1, 1)$, determine span$\{v_1\}$.

12. Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $\mathbb{M}_2(\mathbb{R})$. Determine span$\{A_1, A_2, A_3\}$.

13. Determine the subspace of P_2 spanned by $p_1(x) = 1 + 3x$, $p_2(x) = x + x^2$, and decide whether $\{p_1, p_2\}$ is a spanning set for P_2.

14. Show that $v_1 = (2, -1), v_2 = (3, 2)$ span $\mathbb{R}^2$, and express the vector $v = (5, -7)$ as a linear combination of v_1, v_2.

15. Show that $v_1 = (-1, 3, 2)$, $v_2 = (1, -2, 1)$ and $v_3 = (2, 1, 1)$ span $\mathbb{R}^3$, and express $v = (x, y, z)$ as a linear combination of v_1, v_2, v_3.

16. Prove that if S and S' are subsets of a vector space V such that $S \subseteq S'$, then span(S) is a subset of span(S').

17. Prove that span$\{v_1, v_2, v_3\}$ = span$\{v_1, v_2\}$ if and only if v_3 can be written as a linear combination of v_1 and v_2.

18. Suppose $\{v_1, v_2, ..., v_n\}$ spans a vector space V, and suppose that v_n is a linear combination of $v_1, v_2, ..., v_{n-1}$. Show that $\{v_1, v_2, ..., v_{n-1}\}$ spans V as well.

19. Check which of the following are true or false and justify:
   (a) If S is a spanning set for a vector space V and W is a subspace of V, then S is a spanning set for W.
   (b) The linear span of two vectors in $\mathbb{R}^3$ is a plane passing through origin.
   (c) Every vector space V has a finite spanning set.

20. Prove or give a counter example: if U_1, U_2, W are subspace of V such that $U_1 + W = U_2 + W$ then $U_1 = U_2$.

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