MA 102 : Linear Algebra and Integral Transforms
Tutorial Sheet - 2
Second Semester of Academic Year 2019-2020

---

1. Let $L(S) = \text{span}(S)$ denote the subspace spanned by S in a vector space V, then show that:
   (a) $L(S)$ is the subspace containing S.
   (b) if $S \subseteq T \subseteq V$ and T is a subspace of V then, $L(S) \subseteq T$. (i.e. $L(S)$ is the smallest subspace containing S).

2. Show that the set S containing all $n \times n$ symmetric and anti-symmetric matrices forms a spanning set for the vector space $M_n(\mathbb{R})$ of all $n \times n$ real matrices.

3. Give argument to show that the vector space of all real valued functions on $\mathbb{R}$ is spanned by the set containing all even and odd functions.

4. Write the vector $v = (a, b, c) \in \mathbb{R}^3$ as linear combination of the vectors $u_1 = (1, 2, 0), u_2 = (-1, 1, 2)$ and $u_3 = (3, 0, -4)$.

5. Examine in each case which of the following sets are L.I. over $\mathbb{R}$:
   (a) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\}$ in $M_2(\mathbb{R})$.
   (b) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(\mathbb{R})$.
   (c) $\{(1, -1, 2), (2, 0, 1), (-1, 2, -1)\}$ in $\mathbb{R}^3$.

6. Check the following sets $\{f, g\}$ for linear independence, considered as subsets of the vector space of all real valued functions on $\mathbb{R}$, where:
   (a) $f(x) = x$, $g(x) =| x |$.
   (b) $f(x) = cos(x), g(x) = sin(x)$.
   (c) $f(x) = e^{rx}, g(x) = e^{sx}$ for r not equal to s.

7. Let A be 3×3 matrix and let
   $v = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. Suppose that $Av = v$ and $Aw = 2w$.
   Then find the vector $A^5 \begin{pmatrix} -1 \\ 8 \\ -9 \end{pmatrix}$.

8. Let $S = \{(1 + i, 2i, 2), (1, 1 + i, 1 - i)\} \subset \mathbb{C}^3$. Check the linear independence of S over $\mathbb{R}$.

9. Is the set S considered in problem 7 L.I. over $\mathbb{C}$?

10. Let $v_1 = (a, b, c), v_2 = (d, e, f), v_3 = (g, h, i)$ be any three vectors in $\mathbb{R}^3$. Show that the set $\{v_1, v_2, v_3\}$ is L.D. iff there exists a non zero vector $x = (x_1, x_2, x_3)$ s.t. $Ax = 0$, where A is the matrix $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$.

11. Let V be the vector space and $S_1 \subseteq S_2 \subseteq V$ be any two subsets, then prove that:
    (a) If S_2 is linearly independent set then so is S_1.
    (b) If S_1 is linearly dependent set then so is S_2.

12. Prove that any set S in a vector space V containing the 0 vector is linearly dependent.

13. Let S be the L.I. subset of a vector space V and $v \notin S$. Prove that $S \cup \{v\}$ is L.I. $\iff$ $v \notin span(S)$.

14. Find the value(s) of h for which the following set of vectors
$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} h \\ 1 \\ -h \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 2h \\ 3h+1 \end{pmatrix}$
are linearly independent.

15. Let $\{v_1, v_2, v_3\}$ be a basis of vector space V. Show that the set $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also a basis of V.

16. Show that $(1, 4)$ and $(0, 1)$ form a basis of $\mathbb{R}^2$ over $\mathbb{R}$.

17. Find a basis and hence give the dimension of each of the following vector space containing all $n \times n$ real matrices which are:

    (a) Diagonal.
    (b) Anti-symmetric.
    (c) Having trace zero.
    (d) Upper triangular.

18. Give example of a set V which forms vector space over both $\mathbb{R}$ and $\mathbb{C}$ and both having different dimensions. Is this always true for any such vector space V?

19. Find a basis and dimension of following subspace S of vector space of polynomials $P_n(\mathbb{R})$, where:

    (a) $S = \{p(x) \in P_n(\mathbb{R}) | \;\; p(0) = 0\}$.
    (b) $S = \{p(x) \in P_n(\mathbb{R}) | \;\; p(x)$ is an odd function $\}$.
    (c) $S = \{p(x) \in P_n(\mathbb{R}) | \;\; p(0) = p''(0) = 0\}$.

20. Check whether the vector space $V = P(t)$ of all real polynomials over $\mathbb{R}$ is finite dimensional or not.

21. Find a basis and dimension for the subspaces W_1 and W_2 of $\mathbb{R}^5$, where:

    (a) $W_1 = \big\{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 \mid a_1 - a_3 - a_4 = 0\big\}$.
    (b) $W_2 = \big\{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 \mid a_2 = a_3 = a_4, a_1 + a_5 = 0\big\}$.

22. Suppose M is an $n \times n$ upper-triangular matrix. If the diagonal entries of M are all non-zero, then prove that the column vectors are linearly independent. Does the conclusion hold if we do not assume that M has non-zero diagonal entries?

23. Let V and W be following subspaces of $\mathbb{R}^4$:
$V = \{(a, b, c, d) \mid b - 2c + d = 0\}$, $W = \{(a, b, c, d) \mid a = d, \; b = 2c\}$. Find bases and the dimensions of V, W and $V \cap W$. Hence prove that $\mathbb{R}^4 = V + W$.

⋆⋆⋆⋆⋆ End ⋆⋆⋆⋆⋆