MA 102 : Linear Algebra and Integral Transforms
Tutorial Sheet - 4
Second Semester of Academic Year 2019-2020

---

1. Determine which of the following mappings are linear.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ yz \end{bmatrix}$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+1 \\ y \\ z \end{bmatrix}$.

   (c) Transpose mapping $T : \mathbb{R}^{m\times n} \to \mathbb{R}^{n\times m}$, $T(A) = A^t$.

   (d) Trace mapping $tr : \mathbb{R}^{m\times n} \to \mathbb{R}$, $tr(A) = trace(A)$.

   (e) The evaluation mapping $\varepsilon_u : \mathbb{R}[x] \to \mathbb{R}$, $u \in \mathbb{R}$, defined by $\varepsilon_u(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1u + \cdots + a_nu^n$, where $\mathbb{R}[x]$ is the set of all polynomials over $\mathbb{R}$.

   (f) $T : \mathbb{R}[x] \to \mathbb{R}^\infty$, defined by $T(a_0 + a_1x + \cdots + a_nx^n) = (a_0, a_1, \cdots, a_n, 0, 0, \cdots)$, where $\mathbb{R}^\infty$ is the set of all real sequences.

2. Find out whether the following statements are true or false.

   (a) The differential mapping $\mathcal{D} : \mathcal{C}^1(\mathcal{I}) \to \mathcal{C}^0(\mathcal{I})$, defined by $\mathcal{D}(f(x)) = f'(x)$ is injective, where $\mathcal{I}$ is an open interval in $\mathbb{R}$.

   (b) The mapping defined in Q.1(f) is surjective but not injective.

   (c) The evaluation mapping is surjective.

   (d) The trace mapping is injective but not surjective.

3. Find Null space and Range space of the following mappings.

   (a) $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S\begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \end{bmatrix}$

   (b) $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T\begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ . \\ . \\ . \\ x_n - x_{n-1} \end{bmatrix}$

4. Let T be a linear operator on a vector space V, let $v \in V$ and let m be a positive integer such that $T^m v = 0$ and $T^{m-1}v \neq 0$. Then show that $v, Tv, ..., T^{m-1}v$ are linearly independent.

5. Consider the vector space $\mathbb{P}_2(x)$ of polynomials with real coefficients and of order at most 2. Find the rank and nullity of the following linear transformation $T : \mathbb{P}_2(x) \to M_{2\times 2}(\mathbb{R})$ defined by $T(p(x)) = \begin{bmatrix} p(1) - p(2) & 0 \\ 0 & p(0) \end{bmatrix}$.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through x-axis and then reflects points through the line $y = x$. Find the mapping T.

7. Find a 2×2 singular matrix B that maps $(1,1)^t$ to $(1,3)^t$.

8. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.
   (a) Prove that if $dim(V) < dim(W)$, then T cannot be onto.
   (b) Prove that if $dim(V) > dim(W)$, then T cannot be one-to-one.

9. Find a linear map $F : \mathbb{R}^3 \to \mathbb{R}^4$ whose image is spanned by $(1,2,0,-4)$ and $(2,0,-1,-3)$.

10. (a) Give an example of linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$.
    (b) Give an example of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $N(T) = R(T)$.

11. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (1,0,0), T(1,1,0) = (1,1,1), T(1,1,1) = (1,1,0)$. Find $T(x,y,z),\ Ker(T),\ R(T)$. Prove that $T^3 = T$.

12. Let V be a finite dimensional vector space and $S, T \in L(V)$. Show that if ST is identity operator, then so is TS. Give an counter example showing that the given statement may not be true for infinite dimensional vector spaces.

13. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and let $S : \mathbb{R}^2 \to \mathbb{R}^3$. Show that ST is not injective.

14. Consider the matrix mapping $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that $T(X) = AX$, where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$.
    Find a basis and the dimension of the image of T as well as of the kernel of T.

15. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,1,4)$. What is $T(8,11)$?

$\star\star\star\star\star$ End $\star\star\star\star\star$