MA 102 : Linear Algebra and Integral Transforms
Tutorial Sheet - 5
Second Semester of Academic Year 2019-2020

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1. Suppose the x and y axes of $\mathbb{R}^2$ are rotated counter-clockwise $45°$ so that the new x and y axes are along the line $y = x$ and $y = -x$, respectively. Then
   (a) Find the change of basis matrix P.
   (b) Find the co-ordinates of the point $(5, 6)$ under the given rotation.

2. Consider the linear transformation T on $\mathbb{R}^2$ defined by $T(x, y) = (2x - 3y, x + 4y)$ and the bases $E = \{(1, 0), (0, 1)\}$ and $S = \{(1, 3), (2, 5)\}$.
   (a) Find the matrix A representing T relative to the bases E and S.
   (b) Find the matrix B representing T relative to the bases S and E.
   (c) How are the matrices A and B related?

3. Consider the vector space $\mathbb{P}_3(x)$ of polynomials with real coefficients and of order at most 3. The differential operator $\mathcal{D}$ is a linear operator on $\mathbb{P}_3(x)$. Find the matrix representing $\mathcal{D}$ with respect to basis $\mathcal{B} = \{1 + x, x + x^2, x^2 + x^3, x + x^3\}$.

4. Let V be an n-dimensional vector space over $\mathbb{R}$ and let $T \in L(V)$, such that $(T - \lambda I)^n = 0$, $\lambda \in \mathbb{R}$ and $(T - \lambda I)^{n-1} \neq 0$. Prove that there is a basis $\mathcal{B}$ such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}.$$

5. Let T be a linear operator on a finite dimensional vector space V over $\mathbb{R}$. Prove that if the matrix representation of T with respect to all bases of V is the same, Then $T = \alpha I$ for some $\alpha \in \mathbb{R}$.

6. Prove that if $T \in L(V)$, $dim(V)$ is finite and rank $T = 1$, then $\det(I + T) = 1 + trT$.

7. Prove that the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

8. (a) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.
   (b) Prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

9. Find out whether the following statement is true or false.
   (a) Any two eigenvectors are linearly independent.
   (b) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(c) The sum of two eigenvectors of a linear operator T is also an eigenvectors of T.

(d) Similar matrices always have same eigenvalues.

(e) Similar matrices always have same eigenvectors.

(f) If λ is characteristic root of the matrix A, show that $k + \lambda$ is a characteristic root of the matrix $A + kI$ where I is an identity matrix.

10. If $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, verify Cayley-Hamilton theorem. Hence find A^{-1}.

11. Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.

12. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is diagonal matrix.

(a) $V = \mathbb{R}^2$ and $T(a, b) = (-2a + 3b, -10a + 9b)$.

(b) $V = \mathbb{P}_2$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$.

(c) $V = M_{2\times 2}$ and $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$

(d) $V = M_{2\times 2}$ and $T(A) = A^t + 2.tr(A).I_2$.

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